GEOMETRIC ASPECTS IN MATHEMATICAL FOUNDATIONS OF CARTOGRAPHY

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Cartographic projection represents the relationship between the reference surface of the Earth and its map image. In essence, it has a geometric and mathematical character. In this paper, the emphasis is on the presentation of geometric aspects in the selection and creation of cartographic projection, namely the shape of the reference surface of the Earth, geometric characteristics of the projected area, curves on reference surfaces and image shape requirements of projected elements. These aspects are included, although often hidden in the university textbook "Mathematical Foundations of Cartography", in the creation of which I applied my mathematical logic, geometric eyes and cartographic heart.

Reference surfaces

The reference surface of Earth is the geometric surface that approximates the surface of the Earth's body. In mathematical cartography, a reference sphere and a reference ellipsoid are used, which is a oblate ellipsoid of revolution whose axis is the earth's axis. The choice of the reference surface of the Earth and its parameters greatly influences the values of distortions in the cartographic projection. We define geographical coordinates on the Earth reference surface :

- ellipsoidal latitude φ : $\varphi \in \langle -90°, 90° \rangle$, **spherical latitude** U : $U \in \langle -90°, 90° \rangle$,

- **ellipsoidal longitude** λ: $\lambda \in \langle -180°, 180°)$, **spherical longitude** λ: $\lambda \in \langle -180°, 180°)$.

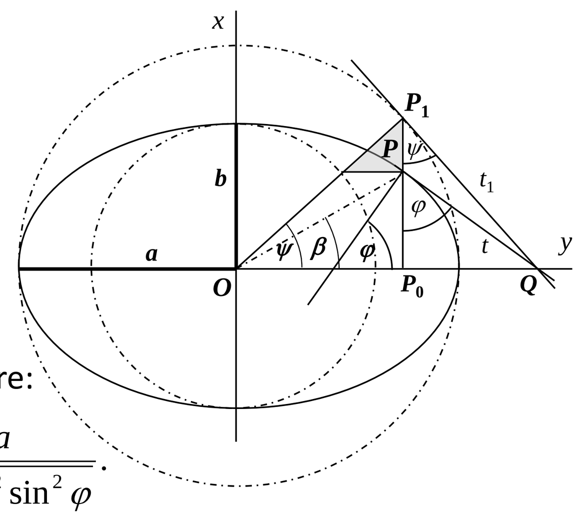
Coordinates derived from them using a ***Concentric Circle Method***:

- Geocentric latitude β:

$$tg\beta = (1-e^2)\ \mathrm{tg}\,\varphi$$

- **Reduced latitude** ψ:

$$tg\psi = \sqrt{1-e^2}\ \mathrm{tg}\,\varphi$$



Using differential geometry they are **the principal radii** specified at defined point ellipsoid curvature:

$$M = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}},\quad N = \frac{a}{\sqrt{1-e^2\sin^2\varphi}}.$$

Cartographic projection and distortions

Cartographic projection defines mathematical relation between geographic coordinates of corresponding points on two reference surfaces or between geographic coordinates of the point on the reference surface and planar coordinates of its image in the plane.

From the point of view of algebraic geometry, it is a relationship between two linear manifolds.

Classification of cartographic representations is realized using 3 criteria.

1st criterion - distortions:

a, *equidistant projections* – lengths of a set of curves are preserved
b, *equal-area (equivalent) projections*– areas are preserved
c, *conformal projections* – angles are preserved
d, *compensational projections* – angular and areal distortion are compensated

2nd criterion - projection surface:

a, ellipsoid to the sphere
b, true projections (on the developable surfaces):
- azimuthal projection
- conical projection
- cylindrical projection

c, artificial projections:
- pseudoazimuthal
- pseudoconical
- pseudocylindrical

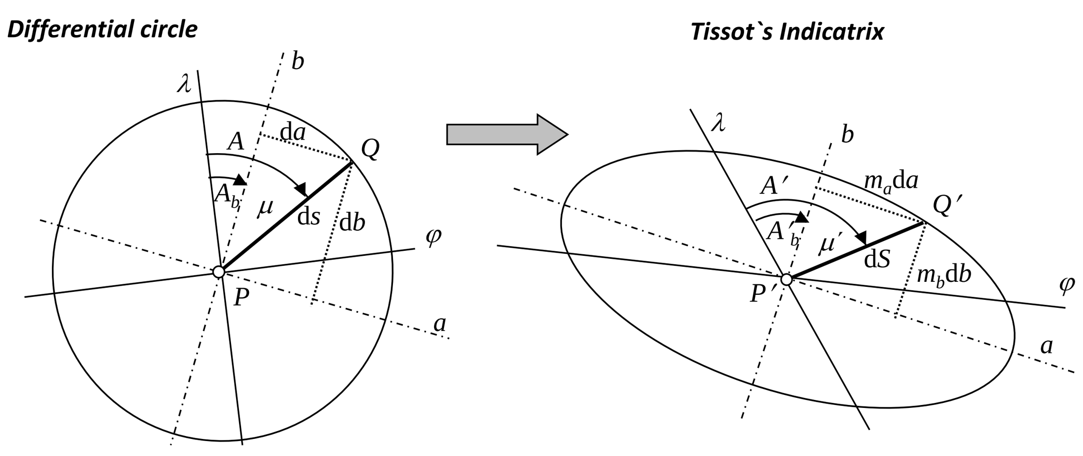
d, polyconic projection
e, polyhedral projection
f, unclassified projection

Classification of cartographic projections on developable surface by 3rd criterion - position:

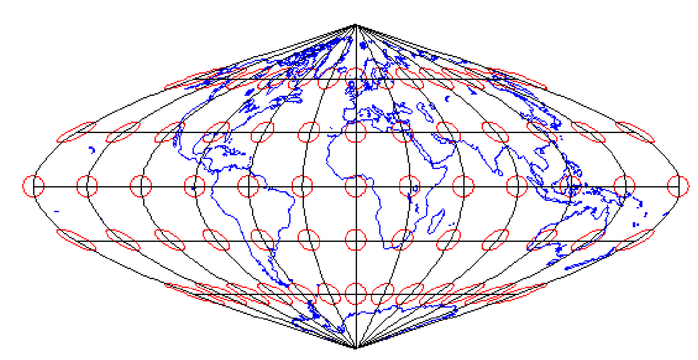
a, normal projection (polar)
b, transverse projection (equatorial)
c, oblique projection

The images of elements of reference surfaces are distorted in cartographic projection, there are scale, angular and areal distortion.

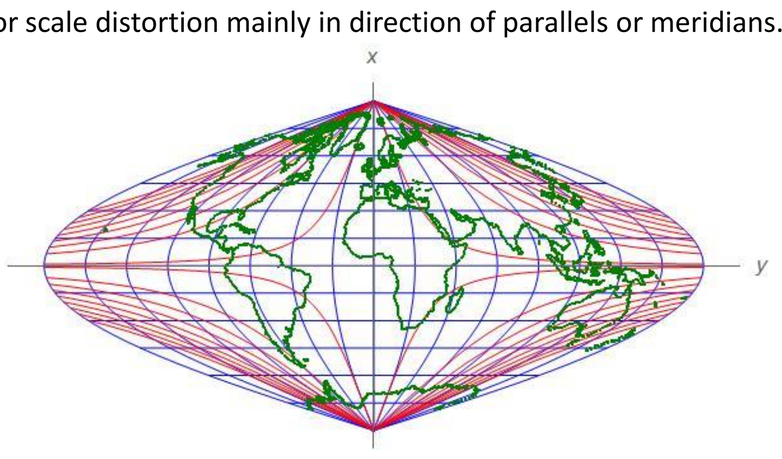
The geometric interpretation of the length distortion at a point is an image of a differential circle with radius ds, which is a ***Distortion Ellipse*** called the ***Tissot`s Indicatrix***.



Directions of axis *a*, *b* with extreme scale distortions, we call ***the principal directions of the mapping***.

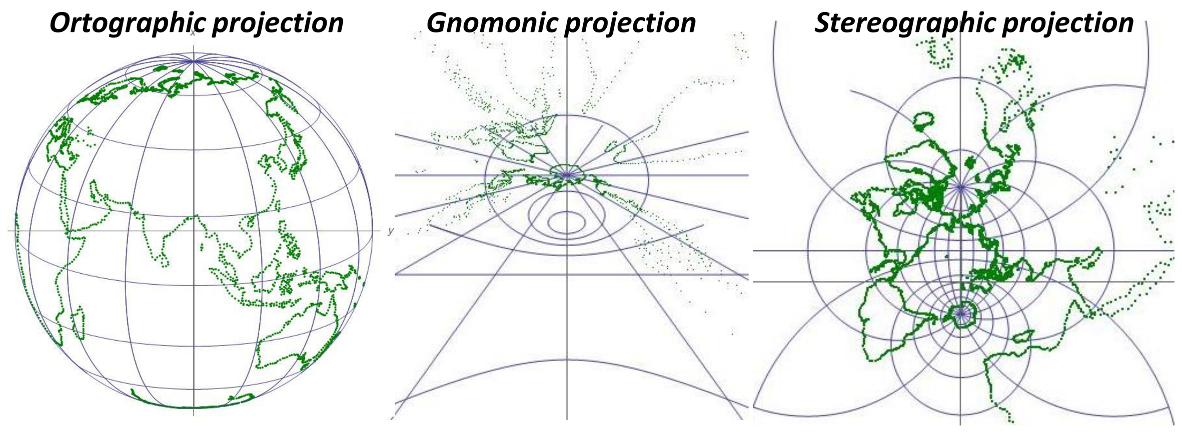


Another way to geometrically represent the length distortion on a map is isometric lines, which are lines with constant distortion. ***Isometric line*** connects points with constant distortion, for scale distortion mainly in direction of parallels or meridians.



Cartographic projections with geometric principle

In history, cartographic representations were created on a geometric basis. ***Azimuthal perspective projections*** (orthographic, stereographic, gnomonic) of the reference sphere into the plane were formulated in ancient Greece and were applied to the construction of maps of the Earth and astronomical map.



Cylindrical perspective projections are also very well known, the names and principles of which are given in the following overview.

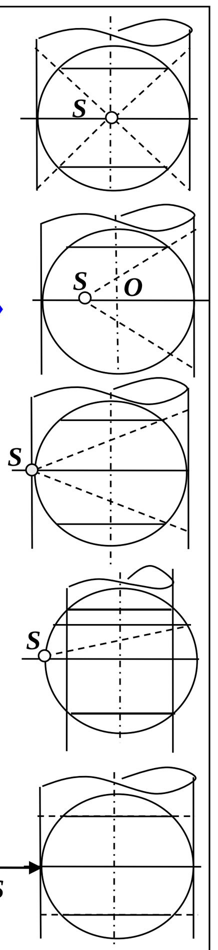
Gnomonic cylindrical projection - on the tangent cylindrical surface, projection center *S* is in the center of the reference sphere.

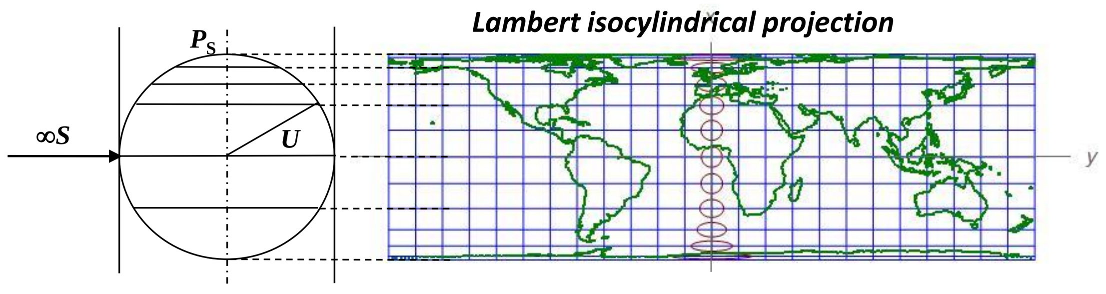
Mercator modified projection - on the tangent cylindric surface, projection center *S* it holds: $|SO|$ = 0,4*R*.

Braun stereographic projection - on the tangent cylindric surface, projection center *S* lies on the reference sphere.

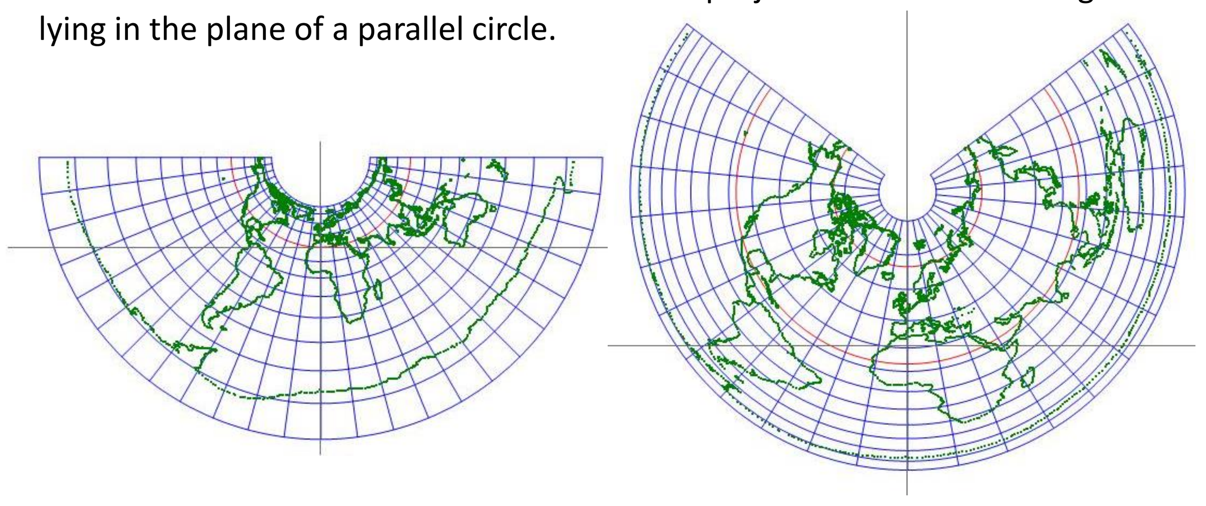
Gall stereographic projection - on the secant cylindric surface ,the intersection of the cylindrical surface and the reference sphere is a parallel circle with U_0 = 45°, projection center *S* lies on the reference sphere.

Lambert isocylindric projection - equivalent cylindrical projection on the tangent cylindric surface, center projection is infinity point ∞S in the equator plane.

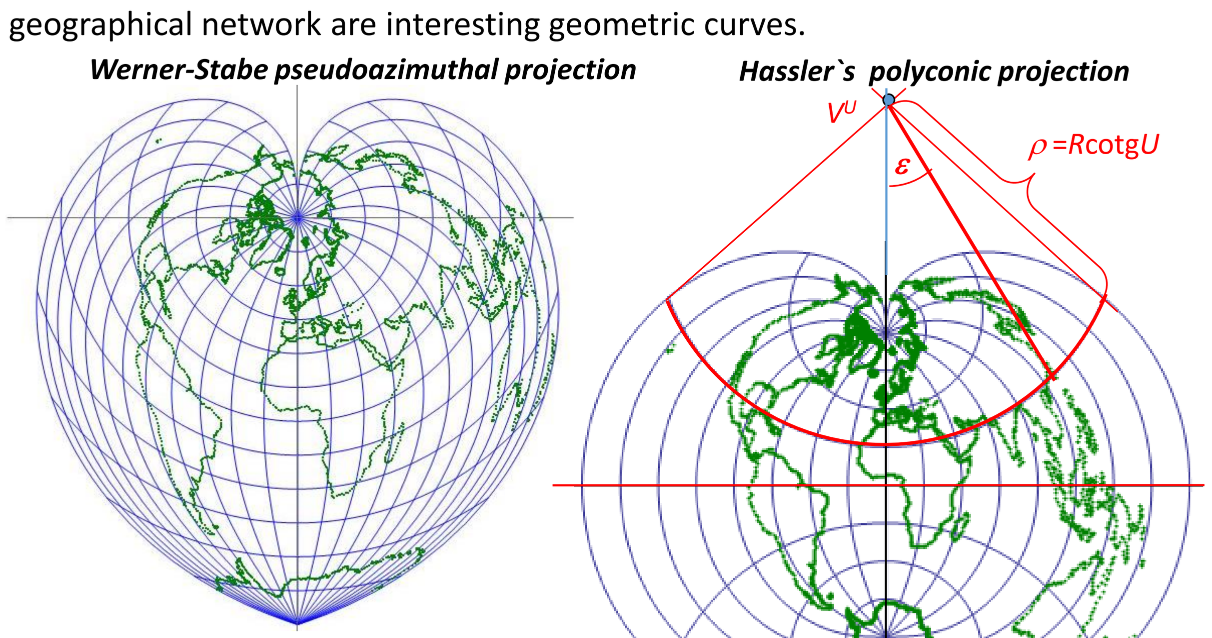


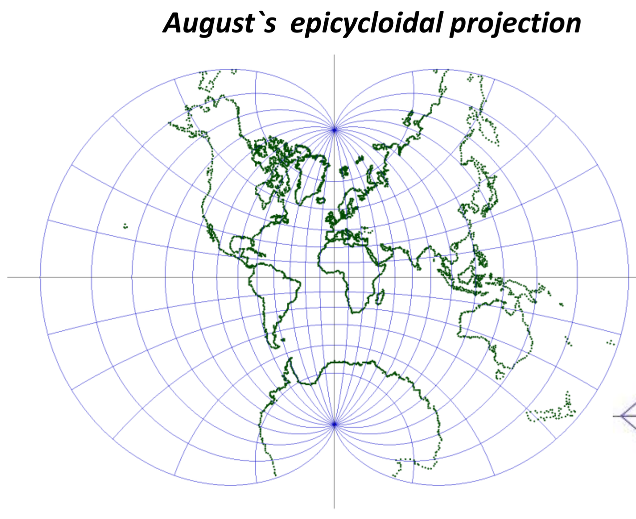


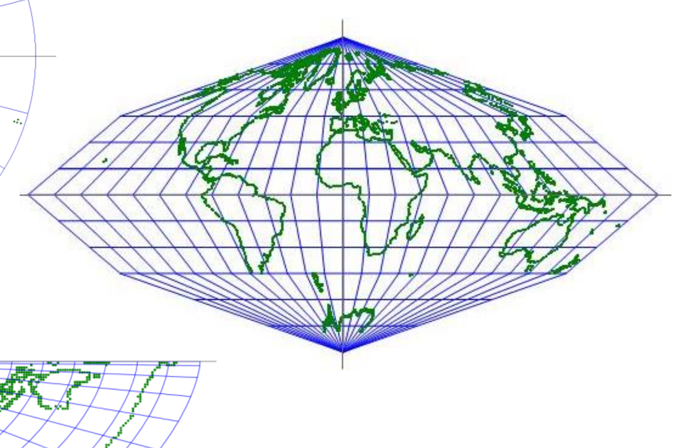
Conical perspective projections have a similar principle as cylindrical projections. The picture shows the image of the geographical network and the boundaries of the continents in two conical projections with a moving center lying in the plane of a parallel circle.

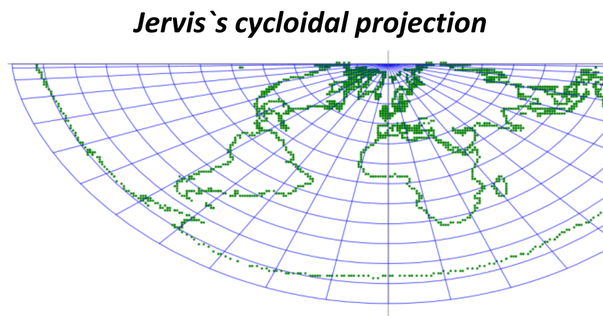


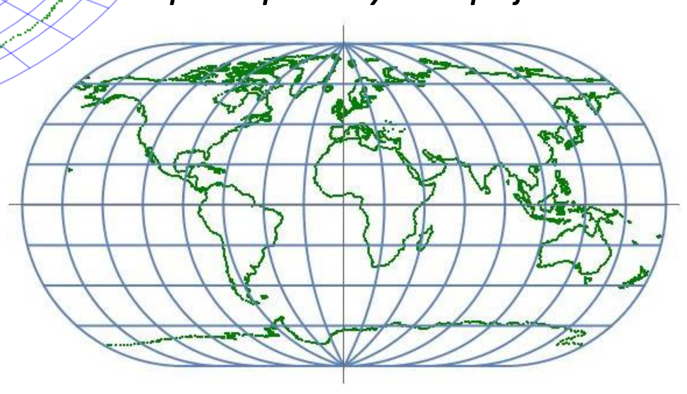
Several cartographic representations are constructable and the image of the geographical network are interesting geometric curves.











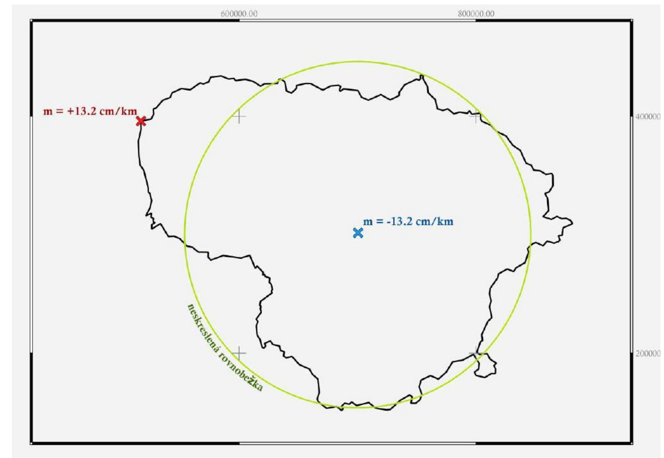
Geometric properties of projected area and selection of cartographic projection

The shape, size and position of the projected area, according to which the type and position of the display area is determined, or their number will be used in the cartographic representation. An overview of the optimal selection of the projection type is given in the Table:

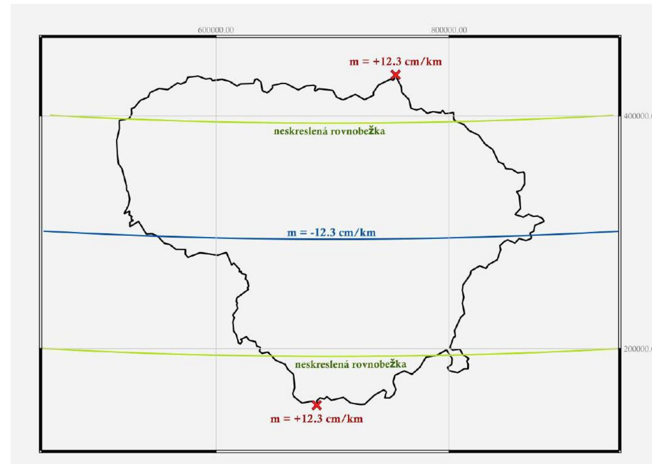
The shape and position of the projected area		*Cartographic projection*	
Circular area	in the pole area	Azimuthal	in polar position
	in the equator region		in transversal position
	in other places		in oblique position
Oblong area	along the equator	Cylindrical	in polar position
	along the earth meridian		in transversal position
	along the orthodrome		in oblique position
	along the earth parallel	Conical	in polar position
	along the cartographic parallel		in oblique position

The influence of these criteria on the extreme length distortion is illustrated in the following figures, where there is a comparison of different types of cartographic projections for the territory of Lithuania (student project).

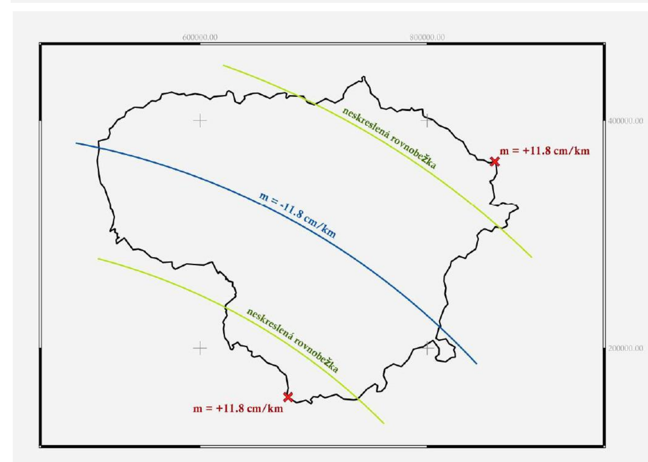
Azimuthal conformal projection in the oblique position,
scale distortion ±13.2 cm/km



Conical conformal projection in the polar position,
scale distortion ±12.3 cm/km



Conical conformal projection in the oblique position,
scale distortion ±11.8 cm/km

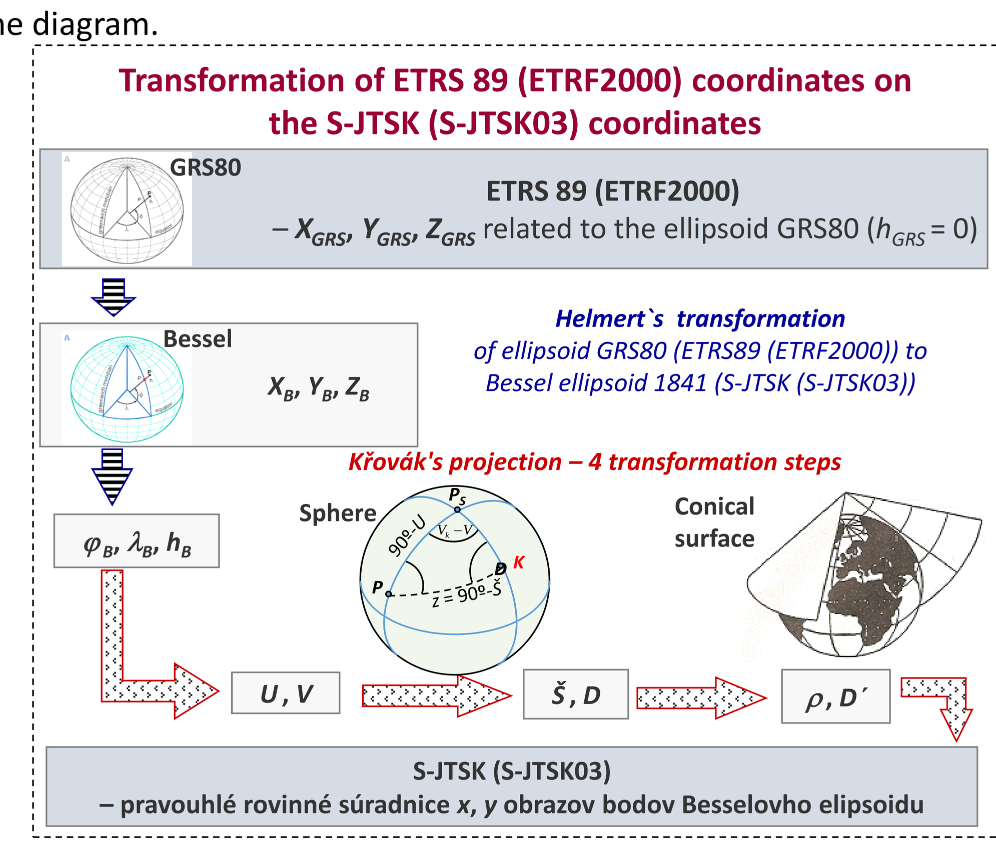


Cartographic projections of Slovakia and transformations between them

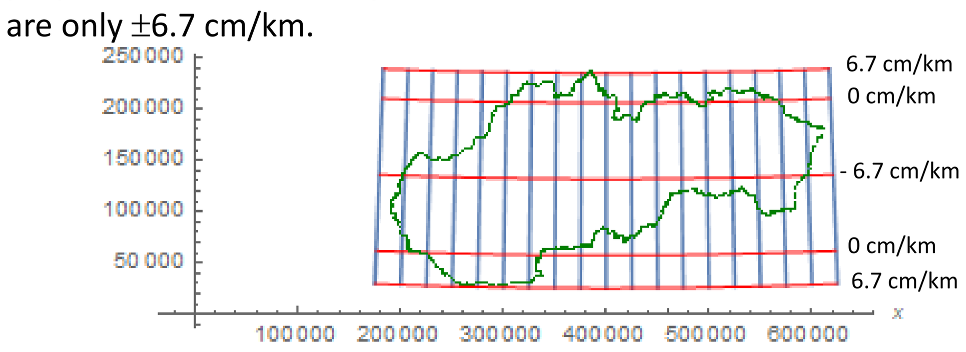
At present and in the near past, several obligatory Geodetic coordinate systems have been used in Slovakia: :

- **S-JTSK (S-JTSK03)**, Bessel ellipsoid, Křovák`s conformal conical projection in the oblique position,
- **UTM** (Zones 33N and 34N), ellipsoid WGS84 with parameters of GRS80, projection Universal Transverse Mercator (conformal cylindrical projection in tansversal position, on the secant cylindrical surface),
- **S-42 (S-42/83/03)**, Krasovskij`s ellipsoid, Gauss-Krüger conformal cylindrical projection in transversal position, on the tangent cylindrical surface,
- **ETRS89 (ETRF2000)**, spatial coordinate system related to the GRS80 ellipsoid.

The transformations between these coordinate systems are based on their geometric factors, namely the geometric parameters of the ellipsoid and the type of cartographic projection. One of the transformations is shown in the diagram.



Křovák's conformal conical projection in the general position applied in the S-JTSK system is not an optimal for the Slovak Republic, extreme length distortions are from -10 to 11 cm/km. In 2010, at the request of the Geodesy, Cartography and Cadastre Authority of the Slovak Republic, Lambert's conformal conical representation in a normal position with parameters for the Slovak Republic was prepared (LCC for SR) and published in (Vajsáblová, 2011). LCC for SR respects the geometric properties of the territory of Slovakia, therefore the maximum distortions are only ±6.7 cm/km.



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