

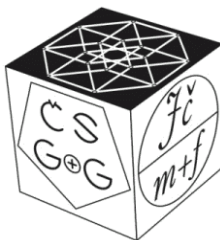
SLOVAK SOCIETY FOR GEOMETRY AND GRAPHICS

34th Symposium on
Computer Geometry SCG'2025



PROCEEDINGS OF THE
11th SLOVAK-CZECH CONFERENCE ON
GEOMETRY AND GRAPHICS 2025

Bratislava, Slovak Republic
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45th Conference on
Geometry and Graphics

CZECH SOCIETY FOR GEOMETRY AND GRAPHICS
OF THE UNION OF CZECH MATHEMATICIANS AND PHYSICISTS

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11th SLOVAK-CZECH CONFERENCE ON GEOMETRY AND GRAPHICS 2025

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TABLE OF CONTENTS

FOREWORDS	7
PLENARY TALKS	9
COCCHIARELLA LUIGI	
<i>Be it pencil or AI: Informing geometry in forming architecture</i>	11
FERDIÁNOVÁ VĚRA, HOLEŠOVÁ MICHAELA	
<i>Historical constructions of ellipses and ovals from the perspective of Franz Sales Meyer and Nicolaus Fialkowski</i>	13
CHALMOVIANSKÁ JANA	
<i>Správa zo stretnutia matematika s deskriptívnou geometriou .</i>	31
SALVI PÉTER (1)	
<i>Aesthetic curve families in computer – aided design</i>	43
ZAMBOJ MICHAL	
<i>Current challenges and perspectives of descriptive geometry .</i>	45
CONTRIBUTED TALKS	51
BAZSO ÁGNES	
<i>Po stopách Feuerbachovej hyperboly: historický prehľad a geometrické vlastnosti</i>	53
BEGANOVÁ JULIANA, KROMKOVÁ STANISLAVA	
<i>Modeling 3D environments for virtual reality</i>	63
GERGELITSOVÁ ŠÁRKA, HOLAN TOMÁŠ	
<i>Složenko. Procvičování prostorové představivosti hrou</i>	69
HLAVOVÁ MARTA, LINKEOVÁ IVANA, PAJEROVÁ NIKOLA	
<i>Staň se na den vědkyní a vymodeluj si obalovou plochu</i>	75
CHALMOVIANSKÝ PAVEL	
<i>Group of isometries of hyperbolic plane</i>	81
CHODOROVÁ MARIE	
<i>Konvexní pětiúhelníky v matematických soutěžích</i>	89

KUBÁT JAKUB, HLAVOVÁ MARTA	
<i>Intersections of surfaces of revolution: Interactive modelling</i>	97
MAKOVNÍK MARCEL	
<i>Note on refinement curve length</i>	103
MARKECHOVÁ IVETA	
<i>A notelet to gyroid: Straight lines invited</i>	109
MOLNÁR EMIL	
<i>On Kártesi type triangle geometry by geometric (Grassmann-Clifford) algebra</i>	117
MOLNÁR MICHAL	
<i>Lie group approach to envelope surfaces</i>	123
PECH PAVEL, BLAŽEK JIŘÍ	
<i>Limits of GeoGebra by solving a problem</i>	133
POKORNÁ BARBORA	
<i>3D pen-supported geometry learning in secondary schools</i>	141
RÜCKSCHLOSSOVÁ TATIANA	
<i>Aplikácia rotačných plôch v architektúre</i>	149
SKRZYPIEC MAGDALENA	
<i>On orthogonal trajectories of isoptics</i>	155
STACHEL HELLMUTH	
<i>Flexible cross-polytopes with two flat poses in three and four dimension</i>	161
VAJSÁBLOVÁ MARGITA, RÜCKSCHLOSSOVÁ TATIANA	
<i>Sundials and gnomonic azimuthal projection</i>	171
VELICHOVÁ DANIELA	
<i>Pairs of inverse curves</i>	185
VOTOUPAL VÁCLAV	
<i>Proměny výuky deskriptivní geometrie na VŠB-TUO</i>	193
LIST OF PARTICIPANTS	199

Forewords

Slovak University of Technology, Institute of Mathematics and Physics at the Faculty of Mechanical Engineering in Bratislava, hosted the 11th Slovak-Czech Conference on Geometry and Graphics held on September 1 – 4, 2025. Conference was the next successful common event of two traditional conferences organized by national societies for geometry and graphics, 34th Symposium on Computer Geometry SCG'2025 in Slovakia and the 45th Conference on Geometry and Graphics in the Czech Republic, and was arranged under the leadership of the Slovak Society for Geometry and Graphics.

42 registered participants from 6 countries – Slovakia, Czech Republic, Austria, Hungary, Poland and Italy attended conference and contributed to its quality scientific programme. The atmosphere of the meeting was as usually very cordial and enjoyable. The rich conference programme included lectures and presentations from various fields of geometry and graphics. Four invited plenary lectures were presented, with one additional invited presentation.

Professor Luigi Cocchiarella from Milan Polytechnic in Italy, former president of the International Society for Geometry and Graphics ISGG, presented in his opening talk “Be it pencil or AI: Informing Geometry in Forming Architecture” many valuable ideas on the role of geometry in technical education, architecture in particular.

Nice plenary talk on “Aesthetic Curve Families in Computer-Aided Design” was presented by Péter Salvi from Budapest University of Technology and Economics, Hungary. Participants could learn about what we actually mean by aesthetic curves, how this question might be explored by revisiting classical curves generally regarded as beautiful, and about families of aesthetic curve representations, with a deeper look into so-called log-aesthetic curves and their variations. There was also presented the intriguing connection to Archimedean spirals, offering insights into possible applications.

Invited lecture “Historical Constructions of Ellipses and Ovals from the Perspective of Franz Sales Meyer and Nicolaus Fialkowski” introduced to participants historical methods of constructing ellipses and oval shapes. Věra Ferdiánová from University in Ostrava, Czech Republic and Michaela Holešová from University in Žilina, Slovakia, presented Franz Sales Meyer constructions of ellipses and ellipsoids, which were essential for designing more complex forms in architecture and decorative arts, and Nicolaus Fialkowski systematic overview of both classical and lesser-known methods for constructing ellipses, often based on the practical use of circles and arcs. The lecture not only introduced selected methods from the theoretical perspective and through practical demonstrations, but emphasised on their application in education of geometry with potential integration of digital tools, e.g. GeoGebra.

Invited speaker Michal Zamboj from the Charles University in Prague, Czech Republic, described in his interesting invited lecture “Current Challenges and

Perspectives of Descriptive Geometry“ the current problems that descriptive geometry encounters in the field of education, applications, and research. He identified the original motivations as fixed elements accompanying the subject since its foundation. The aim of this lecture was to propose projections of these motivations to the current time into two interrelated projection planes: descriptive geometry as a part of mathematics and as an application in other subjects of not only technical but also general education.

Jana Chalmovianská from the Comenius University in Bratislava, Slovakia, presented her invited lecture as “Report from a meeting between a mathematician and descriptive geometry”, bringing reflections of a mathematician on teaching descriptive geometry to prospective teachers of the subject, offering thus a viewpoint of an outsider, and informing about implementations of the observations into the current transformations of the course at the Faculty of Mathematics, Physics and Informatics.

Submitted 19 contributed talks from applied and pure geometry, graphics and education of geometry are published in this proceedings.

Conference was organized by the Slovak Society for Geometry and Graphics at its official seat, the Institute of Mathematics and Physics, Mechanical Engineering Faculty of the Slovak University of Technology in Bratislava, Slovakia. Social programme included the boat trip on Danube River to the nearby iconic castle Devín, firmly connected with the history of Slavic peoples in the territory of Slovakia. Conference dinner was held in the historic environment of the former Monastery of the Merciful Brothers, today popular Bratislava Flag Ship restaurant, in a relaxing atmosphere with the traditional Slovak cuisine and home-brewed beer.

We would like to invite all interested parties to attend the next joint event of the 46th Conference on Geometry and Graphics and the 35th Symposium on Computer Geometry SCG'2026. This next conference will be held again together in 2026 by representatives of both national societies for geometry and graphics as the 12th Czech-Slovak Conference on Geometry and Graphics in the South Bohemian city České Budějovice, Czech Republic, under the supervision of the Czech Society for Geometry and Graphics.

Let us keep the good tradition of our common meetings deeply rooted in the history.

October 31, 2025

Daniela Velichová
chair of SSGG

Zbyněk Šír
chair of ČSGG

PLENARY TALKS

Be it pencil or AI: Informing geometry in forming architecture

Luigi Cocchiarella

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Abstract. IT has increasingly revealed the dual function of Geometry, that is, as a meta-language supporting the generative processes in design, and, as the structure of that same meta-language.

Keywords: Geometry, architecture, interdisciplinary cooperation, visualization, artificial intelligence

IT has increasingly revealed the dual function of Geometry, that is, as a meta-language supporting the generative processes in design, and, as the structure of that same meta-language.

Parametric modelling by visual programming made us familiar with the visualization of the logical patterns behind geometric configurations and transformations.

As an evolution of CAD modelling, the level of abstraction remained close to that of the pure Geometry anyway. When parameterisation evolved further, new sets of information were digitally integrated in the geometric structures, increasing the semanticity level of the models, on which in turn, they could finally have an impact during the design processes.

Therefore, geometries could even be generated with no direct geometric inputs, or, by operating on other types of data, which was the antechamber of the shape generation by ‘prompts’ via AI.

On the one hand, this new mode reflects that long-desired permeability among languages that is promising in terms of feeding novel ways of interdisciplinary cooperation.

On the other hand, it raises a question about ‘how’ we are expected to think and learn in this new syncretic system of knowledge, independently on the tools used to learn.

Probably, we have to dive more closely, in research, education, and profession, into the profound connections of Geometry with the wide range of disciplinary fields to which it is linked, in other words, into how its purity is essential to the many components of the real realm including architecture, considered in all the scalar extensions of its humanized space, which is a significant part of that real realm.

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Historical constructions of ellipses and ovals from the perspective of Franz Sales Meyer and Nicolaus Fialkowski

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Abstract. This lecture focuses on historical methods of constructing ellipses and oval shapes, as documented in the works of Franz Sales Meyer and Nicolaus Fialkowski. In his renowned Handbook of Ornament, Franz Sales Meyer presents not only ornamental elements but also devotes a section to the constructions of ellipses and ellipsoids, which were essential for designing more complex forms in architecture and decorative arts. Nicolaus Fialkowski, in his comprehensive textbook *Die Zeichnende Geometrie*, offers a systematic overview of both classical and lesser-known methods for constructing ellipses, often based on the practical use of circles and arcs. The lecture will introduce selected methods from both a theoretical perspective and through practical demonstrations, with an emphasis on their application in geometry education and the potential integration with digital tools such as GeoGebra.

Keywords: Nicolaus Fialkowski, Franz Sales Meyer, ellipses, ovals

1 Introduction

In technical practice, the terms oval, ellipse, and oval describing an ellipse are often used interchangeably. Although an exact mathematical definition of an oval in the Euclidean plane is not known, it is usually described as a curve with the following properties.

- differentiable (visually smooth, without sharp breaks),
- simple (does not intersect itself),
- convex (the line segment connecting any two points on the curve lies within the area it bounds),
- closed,
- and has at least one axis of symmetry.

Unlike the oval, an ellipse has a precise mathematical definition as the locus of points whose sum of distances from two fixed foci is constant. Despite their visual similarity, the ellipse differs by its continuously changing curvature. Therefore, although the terms are often confused in technical fields, it is

essential for students—especially in technical disciplines—to understand the key differences summarised in Tab. 1.

Criterion	Oval (general/technical sense)	Ellipse
Definition	Descriptive term for an egg-like shape; often composed of circular arcs.	Set of points with a constant sum of distances from two fixed foci.
Mathematical precision	"Ambiguous, context-dependent."	Precisely defined by an analytical equation.
Curvature	Piecewise constant (composed of arcs with fixed radii).	Smoothly and continuously changing at every point.
Foci	Does not have foci in the sense of an ellipse's definition.	Defined by a pair of foci.
Symmetry	Usually has at least one axis of symmetry.	Always has two axes of symmetry (major and minor).
Relationship	"In geometry, an oval is often defined as a curve similar to an ellipse that is not an ellipse."	"A specific, precisely defined curve."

Tab. 1: Comparison between an oval and an ellipse

We focus on two authors who, in their works, presented many interesting constructions of ovals as well as ellipses that were used in technical practice in the past and remain of interest even today. These shapes are not just decorative — they reflect a time when geometry and aesthetics were deeply intertwined. At the heart of this period stood **Franz Sales Meyer**, who saw ellipses as key to ornamental harmony, and **Nicolaë Fialkowski**, who approached them as powerful tools for geometric thinking and teaching.

Their constructions reveal more than methods — they offer insight into how geometry shaped space, beauty, and learning. And even today, these historical approaches can inspire modern education and design.

2 Nicolaus Fialkowski (1818-1902) and his works

Nicolaus Fialkowski was an Austrian educator, architect, and author of textbooks on technical drawing and descriptive geometry who worked in Vienna during the second half of the 19th century. According to surviving records, during his lifetime he served as an assistant at the Department of Descriptive Geometry and simultaneously as a substitute teacher for technical drawing in the preparatory year at the Imperial and Royal Polytechnic Institute

in Vienna. He was also a professor of mathematics at the Gumpendorfer Realschule secondary school [8].

At the same time, he held the position of full instructor in geometry, architectural design, and related drawing, and he acted as the custodian of the Geometrical and Architectural Teaching Collection (Kustos der geometrischen und architektonischen Lehrmittelsammlung)[7].

In official directories he is listed as residing in Mariahilf, Kirchengasse No. 138, Vienna [7]. He authored more than 30 works, for example (according to historical notes [1] and archival sources [10]):

- Das Dezimalrechnen mit Rangziffern(1863)
- Zeichnende Geometrie (1879/1880/1882)
- Praktische Geometrie (1892)
- Lehrbuch der Geometrie und des Zeichnens oder die geometrische Formenlehre (1864/1872/1882)
- Analyse des Zeichnens nach der Anschauung nebst Angabe einiger neuerdachter Modelk (1856)
- Der Zeichner und Rolorist nebst den dazu gehörigen Zeichenrequisiten und Materialien. Praktische Anleitung [...] für Real- und Gewerbeschulen, wie auch für den Selbstgebrauche für jeden Zeichner, insbesondere für technische Zeichner. Wien: Wendelin 1857
- Lehrbuch der Planimetrie, L Teil (Zweiter Cursus der Geometrie) Wien/Leipzig: Klinckardt 1882 (5. Auflage).
- Lehrbuch der Planimetrie, IL Teil (Dritter Cursus der Geometrie) 1882 (5. Auflage).

Although he was primarily an educator, some sources also mention him as an architect and designer, which is probably related to the fact that after the marriage in 1899 of his daughter Adele (1866–1945) to the much older builder Josef Prokop (1836–1904), he most likely worked in Prokop's firm [8]. This combination of practical experience and theoretical knowledge is reflected in his textbooks, which demonstrate the connection between geometry and technical drawing, as well as architectural construction.

2.1 Zeichnende Geometrie [2]

The book *Zeichnende Geometrie* (3rd edition) is an extensive and meticulously elaborated work whose aim is to provide students of secondary and commercial schools, as well as technicians and technical draftsmen, with a complete collection of feasible and accurate methods for all types of geometric constructions. The author emphasises practicality and precision, thus creating a handbook that connects the theoretical principles of geometry with the needs of technical practice.

The completeness and scope of the book are evident from its rich visual content — on 138 pages of text, it contains approximately 1,800 figures, which represents nearly four times as many as comparable works of the time. Such a vast collection of illustrations arose because the author included not only well-known constructions but also a large number of new ones, based on contemporary research and extensions of geometric methods. The book also incorporates constructions that rarely appeared in other sources but were of great practical importance for technical drawing — for example, the plotting of required shapes within given figures, determining centroids, or the geometric construction of architectural ornaments.

A significant contribution of the work lies in approximately one hundred new constructions created by the author himself, which supplement previously missing areas of geometry. Interestingly, in the second edition from 1872, the author introduced sixty new constructions. Among the most remarkable innovations, according to Fialkowski, are the general construction of angles in degrees and the determination of the magnitude of drawn angles, as well as both general and special methods of angle division, the determination of diagonal points in the construction of an ellipse, and new constructions of ellipses — including the first strictly geometric construction of the so-called *Eilinie* (egg-shaped curve) in two variants. With these new methods, the author not only expands existing knowledge but also strives for a systematic completion and unification of geometric procedures, thereby making his work the richest and most comprehensive of its kind.

Special attention should also be given to the thoughtful layout of the book, which demonstrates the author's careful planning. On one page, the illustrations are placed, while on the opposite page, the corresponding text is printed. This arrangement brings a number of practical advantages, and the author himself highlights this feature in the prologue as a groundbreaking approach. The distance between the text and the diagrams or constructions is minimal, which facilitates orientation and speeds up the search for the relevant information.

Interestingly, Fialkowski also considered the reader's comfort in studying and teaching, noting that this layout makes the learning process more convenient and clearer than in works using traditional woodcut illustrations. Moreover, this arrangement saves space on the working or drawing table, which is essential for practical work with drawing instruments.

Each exercise follows a similar structure (see Fig. 1):

- Numbering and title – each task is numbered and linked to a specific construction (e.g. Tafel 24, Fig. 381...). If it is an original construction by the author, this is indicated with the note (Vom Verfasser) (see Fig. 2).
- Assignment – a clear geometric requirement (e.g. “Divide the angle into three equal parts”).
- Procedure – a step-by-step verbal guide with continuous references to the construction.
- Notes and applications – practical tips, alternatives, and examples of use in practice (e.g. in architecture).

98

2. Fall, mittels Verlängerung der großen Axe.—Es seien (Fig. 841) die beiden Axen AB , CD , somit auch das Rechteck $EFHG$ gegeben. Man ziehe im Rechteck $EFHG$ die beiden Diagonalen, verlängere die AB über E bis CD über C hinaus, mache $JO = BO = \frac{1}{2}AB$, $Kreis \delta$; theile jeden in 3 gleiche Theile, führe aus g und h die Lote auf ab , und aus i und k die Parabeln am und an , welche die Lote in l und m treffen und diese Punkte als Ellipsenpunkte geben. Die Gegenpunkte l' , m' findet man mittels der AO als geführten Parabeln.

1127. Eine Ellipse mittels Tangenten zu konstruieren.
1. Art. (Fig. 847). Es seien a und a' als die beiden Äxen gegeben. — Man bestimme die beiden Brennpunkte F, F' und beschreibe über der großen Äxe aa' einen Kreis; ziehe durch F

§. 34. II. Man verläugere die folgende *Ex CD* (S. 44). Aber L ist Q , so die Seite GF über F hinaus, macht OJ gleich der halben großen $EF = OB$, ferner $BC = BJ$, verjüngt BC mit E , D in L ; und verjüngt D , B bis N , so ist N in E Fußpunkt; die übrigen 3 Fußpunkte finden man, indem man aus O mit dem Radius ON in den Diagonalen Einschnitte macht.

1122. Eine Ellipse aus ihren gegebenen Axen zu konstruiren.

(Fig. 843a.) Man mache $CE = OF$, beschreibe über OF und Seite c des Parallelogramms efg um das Stück $ek = ef$, AB als Durchmesser Kreise, und ziehe die Geraden $OK, OR, OR'' \dots$ führe durch $K, K', K'' \dots$ die $Kas, K'n, K'o \dots$ $\parallel CD$, mache $Kus = Oa, K'u' = Ob, K'u'' = Oc \dots$ so sind $K, u, o \dots$ Gleichpunkte — (Ses. für Davidov in Pola.

1123. Konstruktion der Diagonalmittelpunkte, wenn die beiden konjugierten Durchmesser gegeben sind (oder in einem Parallelogramme).

3. Fall, wenn nur der kleinste gemeinsame Nenner verlangt werden kann. — Es liege AB und CD (S. 148) gegeben. — Man zeichne das Parallelogramm $EFGB$, liege in diesem die Stützgeraden EH und FG , verlängere die DC über C hinaus, erhalte in O die OH , OC , und mache $OC = OH$. — Es sei AC die Gerade, die Punkt K mit A und B verbindet. — Es erfordern die 2 Punkte Z und M also die Hilfsgerade in den Stützgeraden.

Die 2 anderen Diagonalepunkte werden mittels der aus L und M in CD gezogenen Parallelen L_y und M_x gefunden.

1124. 3. Fall. Von der größten tangente Kurze-
meßer verläuft man nach unten.

Sei AB und CD (Fig. 644) gegeben. — Man erhebe
aus A die Gerade AE in einem die Diagonale
 AC gleichenden Winkel, und ziehe die Parallelen
 KH und FG verläuft die AE durch K und G .
 AB in O die AE schneidet, mache $JO = BO$, dann $BK = BF$;
nied man den Punkt K mit C und D verbinde, so erhält
man die Punkte L und M als die Diagonalschnitte. Die an-
dere erhält man mittels der aus L und M gezogenen
Senkrechten LN und MP auf AE .

1125. In einem Trapez die Diagonalen AC und BD so, dass $AC \perp BD$ gilt. Berechne die Fläche des Trapezes, wenn $AC = 10$ und $BD = 12$ sind.

Es sei das Trapez $EFHG$ (Fig. 845) gegeben.

1089. Man ziehe die beiden Diagonalen EH und FG , führe durch ihren Schnittpunkt O die $AB \parallel EF$ und verlängere die AB über B hinaus; erhalte auf AE in O eine Verticthe OD , mache $TO = BO$, dann $BK =$ der Strecke BE und verbinde K mit C und D durch Gerade, so erhält man L und M die beiden Diagonalschnitte. Die LM ist die Parallele zu AC . Man verlange mittelst der I und M von A nach T das Parallelfeld.

1126. Die halbe Kugel mittels 7 Punkte
zu fünfzehn (Zie. 848)

Gegeben ab und cd . Man beschreibe aus c mit ac den Viertelkreis ac und mit cd den Viertel-

kreis af ; theile jeden in 3 gleiche Theile, führe aus g und h die Bote auf ab , und aus i und k die Parallelen zu ab , welche die Bote in l und m treffen und diese Punkte als Ellipsenpunkte geben. Die Gegenpunkte l' , m' findet man mittels der zu ab geführten Parallelen.

1127. Eine Ellipse mittels Tangenten zu konstruiren.

1. Kr. (Fig. 847). Es seien ab und cd als die beiden Kreise gegeben. — Man bestimme die beiden Brennpunkte F , F' und beschreibe über der großen Axe ac einen Kreis; siche durch F die Fe ab , dann dl ab , also $\angle C, l$ wird e eine Tangente und c der Berührungspunkt sein. — Nimmt man nun in der Peripherie des Kreises beliebige Punkte $f, g, h, \dots$ an, und zeichnet aus jedem derselben ein Rechtswinkel in Kreise l , das je 2 Gegenseiten durch die Brennpunkte gehen, so find in jedem solchen die 2 durch den Brennpunkt nicht gehenden Gegenseitigen Tangenten an die zu zeichnende Ellipse.

1128: 2. Hrt. (Fig. 847a). — Man verlängert die Seite c des Parallelogramms $efgh$ um das Stück $ck = cd$.

theile es in dieselbe viele (hier in 4) gleiche Theile, und verbinde diese Theilungspunkte mit 4, wodurch auf og die Punkte 1, 2, 3 entstehen; 31, 22, 13 sind nun Tangenten an die verlangte Ellipse. (Rad. Post.)

1129. In ein gegebenes Quadrat eine Ellipse berührend zu konstruiren.
Es sei $ABCD$ (Fig. 848) das gegebene Quadrat. — Man ziehe in diesem die beiden Diagon-

nalen AC, BD ; beschreibe aus O mit AO einen Kreis, zeichne in diesem ein beliebiges Rechteck, hier $EFGH$, dessen Seiten von den Diagonalen des Quadrates in J, K, L, M geschnitten werden, so hat man JL als die große und KM als die kleine Axe der einzugezeichnenden Ellipse.

1180. In eine gegebene Ellipse ein Quadrat zu beschreiben.
Es sei ABCD (S. 118) die gegebene Ellipse. — Man

Wie bei ABCD (S. 440) das gezeichnete Beispiel. — Man
nehme die große Kreis AC , dann die kleine Kreis BD und ver-
längere letztere oberhalb B etwas bis a ; lasse die Tangen-
tialung BC in C ziehen, und schreibe damit die Ca aus B ;
 E ; verlänge E mit C durch eine Gerade und lasse a die-
ser aus B die Parallele Da , welche die große Kreis AC in
schneidet. Wodurch man zuletzt $aO = mO$, und nicht durch a
4 Punkten, welche gleich die Schnittpunkte des verlangten

1131. In eine Ellipse ein Parallelogramm

(Fig. 850.) Man ziehe die große Axt AC ,
mache $EO = FO$, ziehe durch E und F 2 pa-

mitte $EO = FO$, hier durch E und F 2 parallele Sehnen, hier $GH \parallel JK$, so folgen G, H, J, K als Eckpunkte des verlangten Parallelogramms.

1132. Einen egyptischen (überhöhten) Vogen mittels eines Lineals und Winkelbogens zu zeichnen.

(Fig. 318a.) Es sei die Höhe ad und die Breite cd gegeben. — Man halbiere die cd in b , ziehe ab und bd und beschreibe an ab den Winkel baf so, daß dessen Scheitel auf bd falle, und die Strecke af mit bd schneiden. — Wende auf bd den

und die *Seiten* noch *ad* und *de* fallen. *Wache* *an* dem
Vincal *ist* die *E* *ste* *es* = *ad*, so wie *e* = *ed*; *Schlage* *an*
ist in *e*, *t* und *g* *Stifte*. Wird man das *Vincal* *ist* *so* *hinos*
und *hinos* *g* *Stifte*, das der *Stift* *a* immer *an* *de*, und *e* *an*

und hinzugefügt, daß der Stift φ immer an α , und φ an α bleibt, so beschreibt der Stift α den Bogen $\alpha\beta$. Wird ferner der Winkelhafen so umgekehrt, daß β auf α zu liegen kommt, so erhält man den Bogen $\alpha\gamma$.

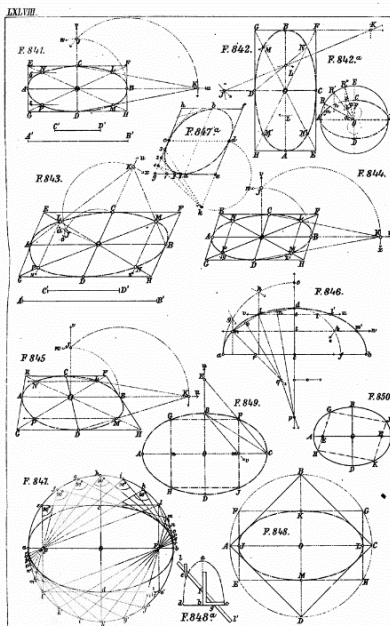


Fig. 1: Example of the book’s layout across a double-page spread [2]

1123. Konstruktion der Diagonalepunkte, wenn die beiden konjugierten Durchmesser gegeben sind (oder in einem Parallelogramme eine Ellipse einzuzichnen). (Vom Verfasser.)

1. Fall, wenn nur der kleine konjugierte Durchmesser verlängert werden kann. — Es seien AB und CD (Fig. 843) gegeben. — Man zeichne das Parallelogramm $EFHG$, ziehe in diesem die Diagonalen EH und FG , verlängere die DC über C hinaus, errichte in O die $Ou \perp CD$, und mache $OJ = OC$, ferner $CK = CJ$. Wird nun der Punkt K mit A und B verbunden, so erfolgen die 2 Punkte L und M als Ellipsenpunkte in den Diagonalen.

Die 2 andern Diagonalepunkte werden mittels der aus L und M zu CD gezogenen Parallelen Ly und Mz gefunden.

Fig. 2: Sample task assignment and description of the construction, marked as the author's [2]

Thanks to its systematic structure, clarity, and practical orientation, Fialkowski's *Zeichnende Geometrie* represents not only a significant milestone in the development of geometry education but also a valuable source for all those engaged in geometric constructions in technical and architectural practice. Owing to the author's diligence and dedication to his field, the book was published in several editions, which he regularly expanded with additional constructions and exercises. In its time, it was a highly respected publication not only in Europe but also abroad; as early as 1886, it was included among the recommended textbooks for instruction in the United States in the Bibliography of Education by G. Stanley Hall and John M. Mansfield of Boston, which is available in the archives of the Library of the University of California, Los Angeles [3].

3 Franz Sales Meyer (1849 – 1927)

Franz Sales Meyer was a German professor, author, poet, and painter from Karlsruhe. During the 19th century, he did not appear as a revolutionary design theorist but rather as a masterly systematiser and educator. His principal contribution lay in the creation of a lasting, practical, and comprehensive framework for the study and application of historical ornament.

From 1866 to 1868, he attended a seminary in Meersburg, which indicates a classical education, and subsequently continued his studies at the Polytechnic in Karlsruhe, where he graduated in 1871 in the field of "industrial art education."

In 1873, he joined the faculty of the Grand Ducal School of Applied Arts (Großherzogliche Kunstgewerbeschule) in Karlsruhe. In 1878, he was appointed a full lecturer and received the title of Professor of Ornamentation. His entire professional life was centred in Karlsruhe, where he also passed away after a short illness on 6 November 1927 and where he had remained active for the rest of his career. Meyer was a multifaceted personality; besides his main professional activities, he was also an accomplished painter and poet,

attesting to his broad cultural outlook. Nevertheless, his primary and enduring legacy was his pedagogical and publishing work in the field of ornament theory.

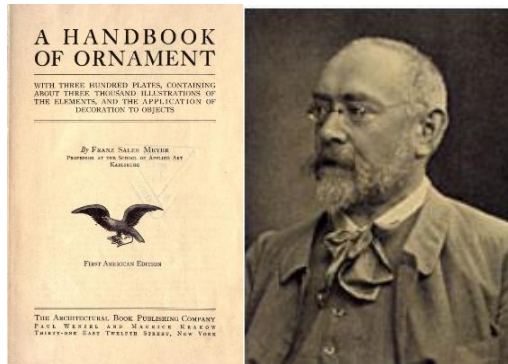


Fig. 3: Left: Page of the book, Right: Picture of Franz Sales Meyer [6]

His most significant work, *Handbuch der Ornamentik*, became an international standard, whose journey began in Germany. It was first published in German in 1888 under the title *Systematisch geordnetes Handbuch der Ornamentik*.

3.1 Handbook of ornaments [6]

The structure of the book represents a carefully designed pedagogical system. In the first part, *Elements of Decoration*, Meyer categorises all ornamental motifs according to their source:

- Geometric elements
- Natural forms (plants, animals, humans)
- Artificial objects (trophies, symbols, etc.)

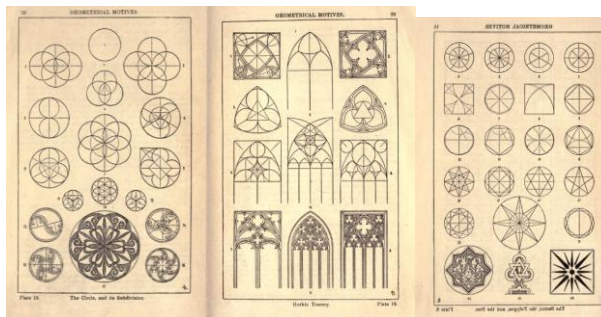


Fig. 4: Example from the first part – Elements of Decoration [6]

In the second part, *Ornament Applied to the Elements*, Meyer classifies ornament based on its structural or compositional purpose:

- Bands (borders)
- Free Ornaments (finials)
- Supports (columns, brackets)
- Enclosed Ornament (panels, fillings)
- Repeating Ornament (patterns)

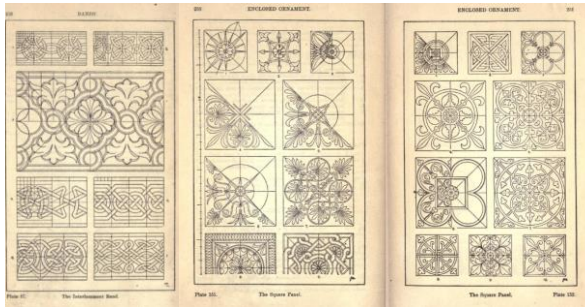


Fig. 5: Example from the second part – *Ornament Applied to Elements* [6]

The third part, *Decorated Objects*, contains case studies in which the principles from the first two parts are applied to real objects. This section demonstrates synthesis in practice and provides students and craftsmen with direct examples of how to decorate specific objects in accordance with their shape, material, and function. The analysed categories include:

- Forms of vases and vessels
- Metal objects (gates, grilles, candlesticks)
- Furniture (chairs, tables, cabinets)
- Frames (for pictures and mirrors)
- Jewellery and personal ornaments
- Heraldry
- Lettering, printing, and bookbinding art

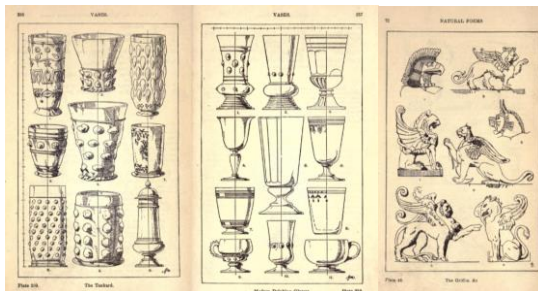


Fig. 6: Example from the third part – *Decorated Objects* [6]

The book contained approximately 3,000 illustrations on 300 plates, making it one of the most comprehensive visual collections of its time. It was translated into several languages and is still sought after as a reference and source of inspiration in the field of design and ornament history, today perceived as a bridge between art history and modern design.

The section most relevant to our study, Geometric Elements, begins with the most abstract forms created through the rhythmic arrangement of points and lines, angle division, and the combination of geometric shapes. It deals with constructions of ellipses (Plate 20, pp. 31, nos. 2–6) and constructions of ovals (Plate 20, pp. 31, nos. 7–12), which can be suitably used for approximating ellipses. Meyer also presents well-known Serlio's oval constructions (Plate 20, pp. 31, nos. 7 and 9), which we have discussed in more detail, for example, in [4].

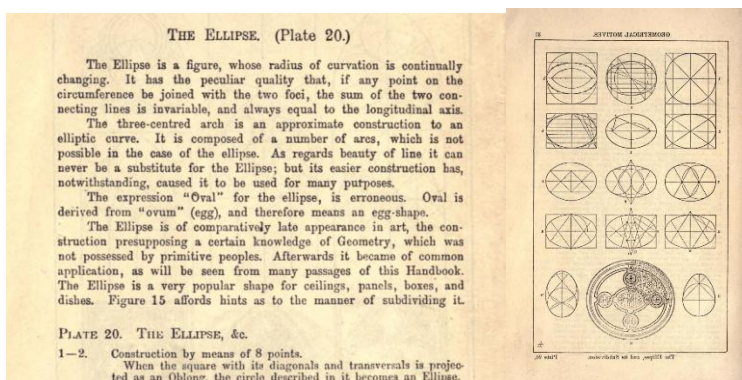


Fig. 7: Example from the first part – Geometric Shapes [6]

4 Historical Constructions of Ellipses and Oval

Both Fialkowski and Meyer understood the ellipse, based on its foci, in the standard way — as the set of all points in a plane for which the sum of the distances from two fixed points (the foci) is constant. Meyer explicitly states:

“It has the peculiar quality that, if any point on the circumference be joined with the two foci, the sum of the two connecting lines is invariable, and always equal to the longitudinal axis.” [6]

Although Meyer was primarily an ornamentalist, he recognised and directly pointed out that the ellipse has a curvature that changes continuously at every point. Fialkowski, in turn, described the fundamental properties of the ellipse, which he illustrated with drawings — particularly the relationships among the foci, tangents, and radii vectors. He also provided a construction for finding the foci when the lengths of the major and minor semi-axes are known.

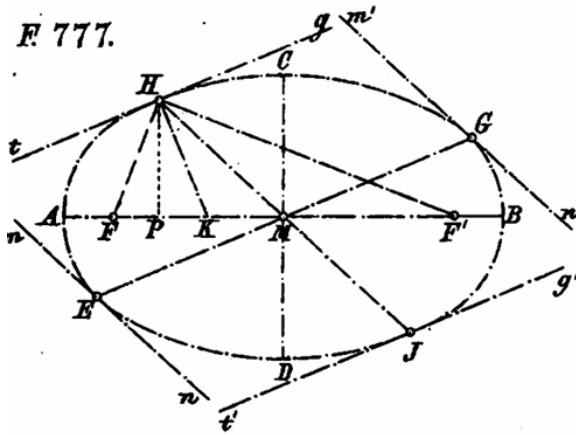


Fig. 8: Illustration of the properties of an ellipse [2]

Both authors also described the ellipse as a special case of the projection of a circle inscribed in a square when projected into a rectangle. Meyer explicitly stated that there exists a construction of eight points of an ellipse: “When a square with its diagonals and transverse lines is projected as a rectangle, the circle within it becomes an ellipse.”[6]

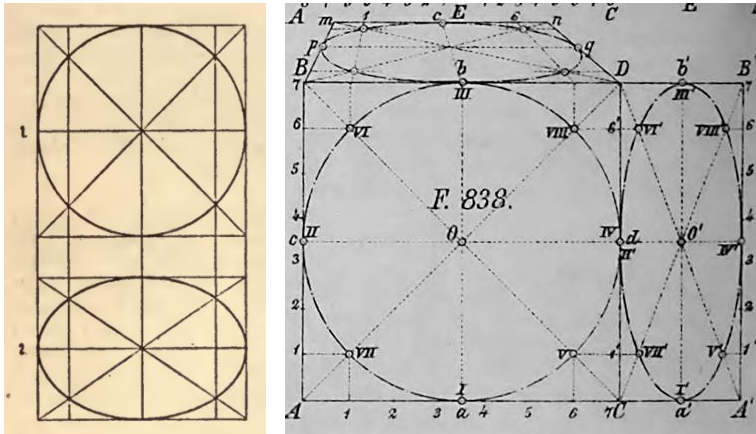


Fig. 9: Construction of the ellipse using eight points.
Left: according to Meyer [6], Right: according to Fialkowski [2]

Since Fialkowski’s book was primarily intended for students of technical schools, it contains around thirty different constructions of the ellipse. Meyer,

on the other hand, presented six basic constructions in his book — mainly those that could be drawn easily and quickly, as he later used ellipses in his ornamental designs.

Regarding the construction of ovals, Meyer explicitly states that calling an ellipse an oval is incorrect, as the term derives from ovum, meaning “egg-shaped curves.” Fialkowski, however, dedicated a separate chapter to approximate constructions of ovals, focusing only on curves resembling the shape of an egg.

If we consider an oval to be a closed convex curve composed of smoothly connected circular arcs, then even these constructions are labelled by Meyer as “Der Korbboogen” [5], which were erroneously translated into English as “ellipsoids” [6], while Fialkowski referred to them as “approximate constructions of the ellipse” [2]. In both cases, we find constructions corresponding to Serlio’s ovals.

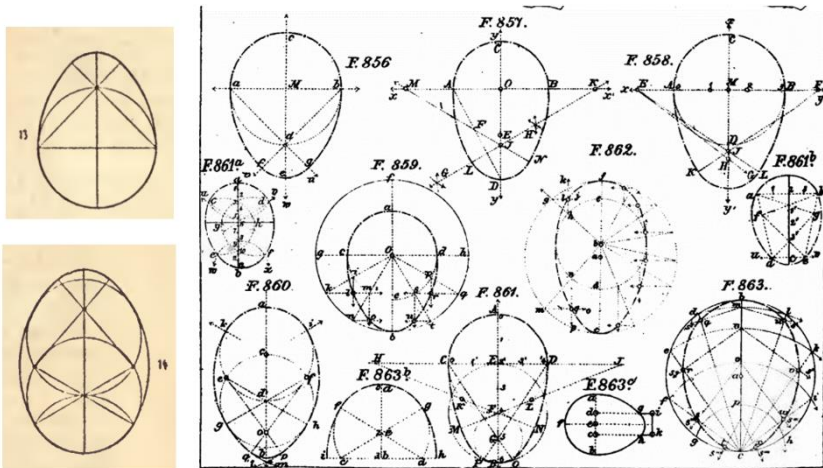


Fig. 10: Examples of ovals Left: according to Meyer [6],
Right: according to Fialkowski [2]

4.1 Fialkowski pp. 92, no.787 [2]

Fialkowski based this construction on the main vertex circle and an auxiliary rectangle in which the ellipse is inscribed. His assumption is that the centre of the ellipse and all its main and minor vertices, labelled A, B, C , and D are known. The author then selects an arbitrary point J on the main vertex circle and constructs a circle centred at the principal vertex B passing through point J . The required point of the ellipse then lies on the diagonal of the rectangle formed from the originally circumscribed rectangle of the circle and the line CL . This is an interesting construction that does not appear in modern textbooks

and is not based on the focal definition of the ellipse. In today's context, with tools such as GeoGebra, it is easy to verify that the determined point M indeed lies on the ellipse.

1057. 2. Art. (Vom Verfasser.)

Man verlängere (Fig. 787) die gegebene große Axe AB über B hinaus, beschreibe aus O mit BO einen Kreisbogen Bu , nehme in demselben einen beliebigen Punkt J an, fälle von diesem auf AB eine Normale JK und schneide die Verlängerung der AB aus B mit BJ in L ein. Verbindet man nun C mit L und F mit K durch eine Gerade, so erfolgt M als Ellipsenpunkt. — Den correspondirenden Punkt N findet man, indem man L mit D und G mit K verbindet. Die andern 2 Punkte P und Q findet man, indem man $K'O = KO$ macht u. s. w.

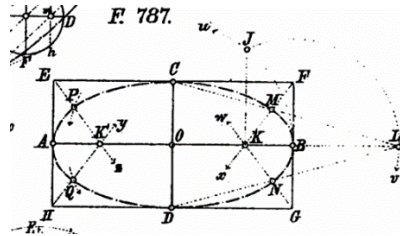


Fig. 11: Construction according to Fialkowski [2]

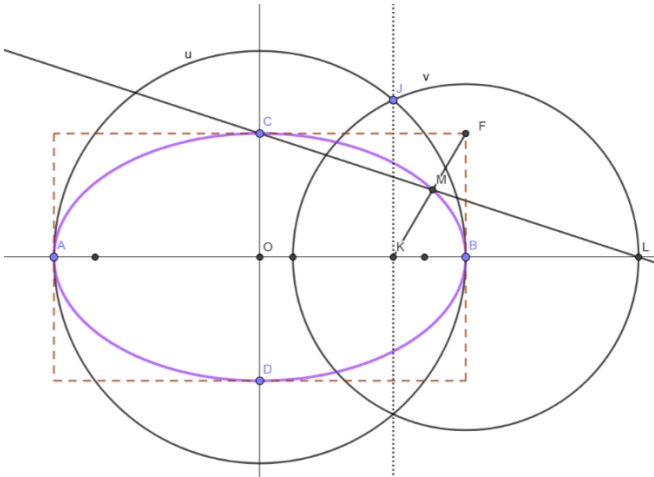


Fig. 12: Construction of the ellipse according to the instructions with verification

Fialkowski pp. 94, no. 809 [2]

Fialkowski most likely adopted construction No. 809 from unknown sources, but it deals with an approximate construction of an ellipse — in our case, an oval — that possesses the property that its eccentricity is equal to the length of the minor semi-axis. To find the centres of the circular arcs that approximate the neighbourhood of the vertices, he used a 3:4 ratio based on the length of the segment AI , which represents half of the remaining distance from the focus to

the vertex. It is a very well-designed construction, as the green circular arcs closely follow the red ellipse. From an analytical point of view, Fialkowski attempted to approximate the ellipse given by the equation

$$\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1,$$

whose radius of curvature at the vertices A and C differs slightly from that of the constructed circles but comes very close — a similarity that, in the technical practice of the time and using contemporary drawing tools, would have appeared nearly identical.

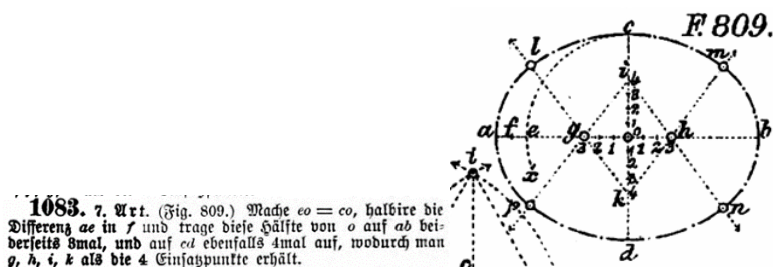


Fig. 13: Fialkowski's approximate construction of the ellipse [2]

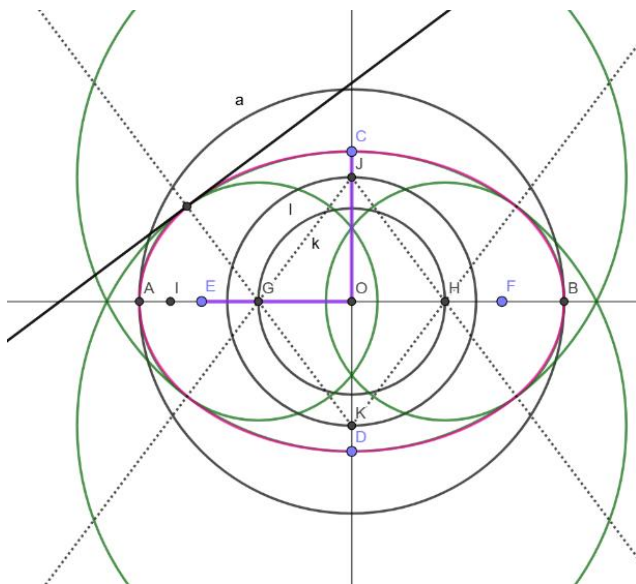


Fig. 14: Construction of an oval using four circular arcs and an ellipse $e = b$

4.2 Meyer Plate 20, pp. 31, no. 8 [6]

This is a modified version of the well-known Serlio construction (see [9]), in which three circles are drawn with radii equal to half the length of the major semi-axis of the given ellipse. The second arc of the approximating oval has its centre at point K , as shown in Fig. 16. Meyer placed the centre of the second circle so that the triangle S_1S_2P is equilateral. Therefore, this construction is suitable for ellipses whose ratio of the major to minor axis is

$$a/b = (\sqrt{3} + 3)/3,$$

and the ratio of the radii of the circles defining the oval is 1:3.

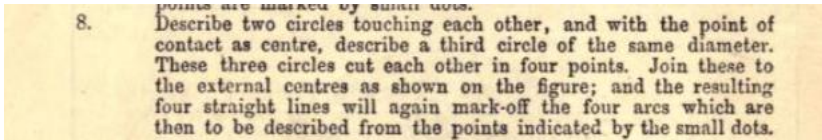


Fig.15: Description of the construction from the book [6]

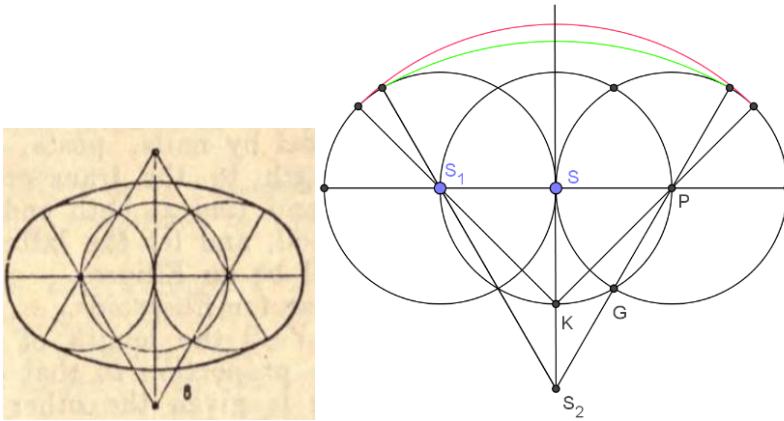


Fig. 16: Left: Meyer's construction in the book [6], Right: Meyer's construction (green colour), Serlio's construction (c) (red colour)

4.3 Meyer Plate 20, pp. 31, no. 10 [6]

We construct a line segment AC and apply the length $a-b$ from the point C to this line segment and we get a point Q . We mark the center of the line segment AQ as a point L . We draw the perpendicular to the line AC through the point L .

This line intersects the major axis at a point S_1 and the minor axis at a point S_2 . See Fig. 18.

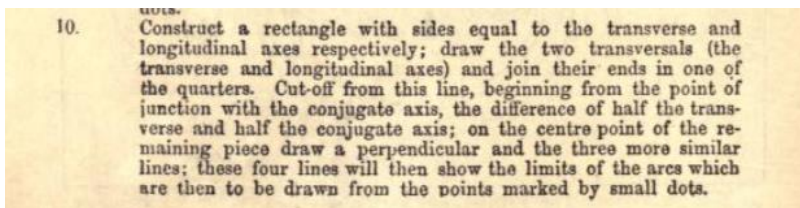


Fig. 17: Description of the construction from the book [5]

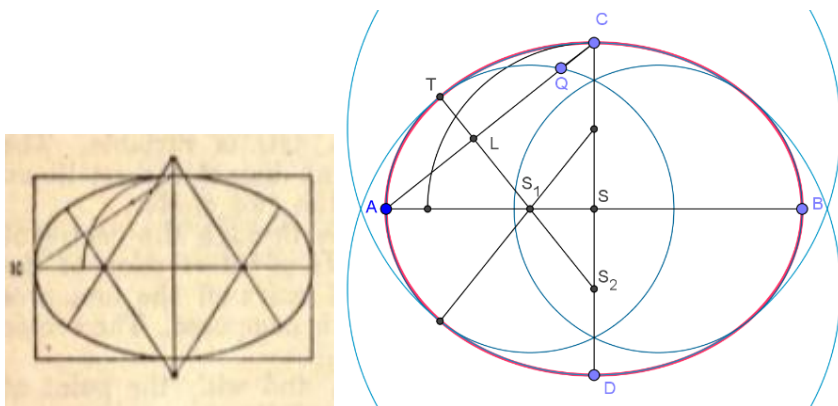


Fig.18: Left: Meyer's construction in the book [6],
Right: Meyer's construction (blue colour), ellipse (red colour) [4]

4.4 Meyer Plate 20, pp. 31, no. 11 [6]

We take half the difference between the semi-major axis and the semi-minor axis of the ellipse. From the center S we apply this distance three times to the major axis and four times to the minor axis.

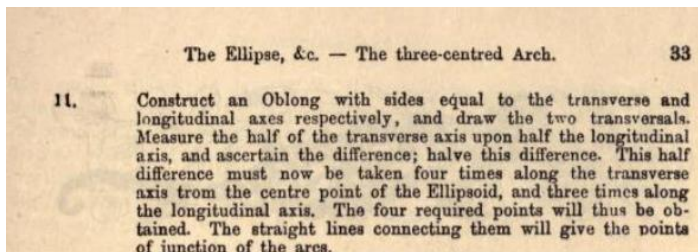


Fig. 19: Description of the construction from the book [5]

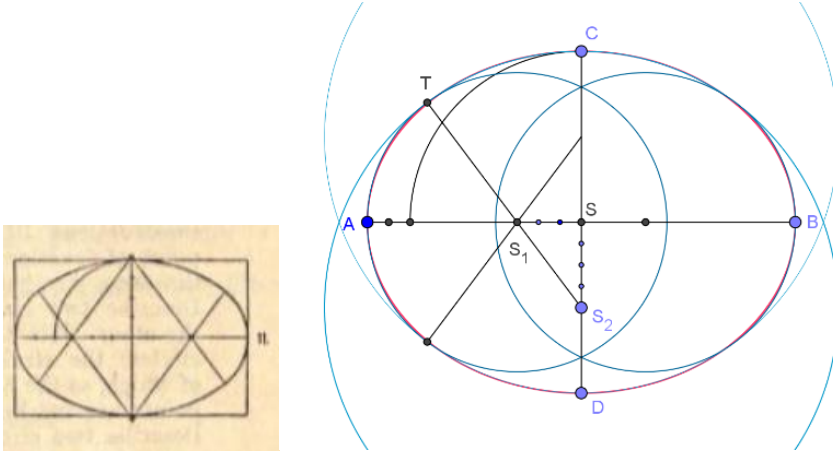


Fig. 20: Left: Meyer's construction in the book [6],
Right: Meyer's construction (blue colour), ellipse (red colour)

4.5 Meyer Plate 20, pp. 31, no. 12

Let us consider an ellipse with a given major and minor axis. We construct its osculating circles with radii $r_1 = |AS_1| = \frac{b^2}{a}$, $r_2 = |CS_2| = \frac{a^2}{b}$. Their centres are S_1, S_2 , as shown in the Fig. 22. Next, we draw circles centred at points S_1, S_2 with a radius of $(r_2 - r_1)/2$. The intersection of these circles determines point O_1 . The third circle, which smoothly connects the hyperosculating circles, is the one shown in green in the Fig. 22 and has its centre at point O_1 .

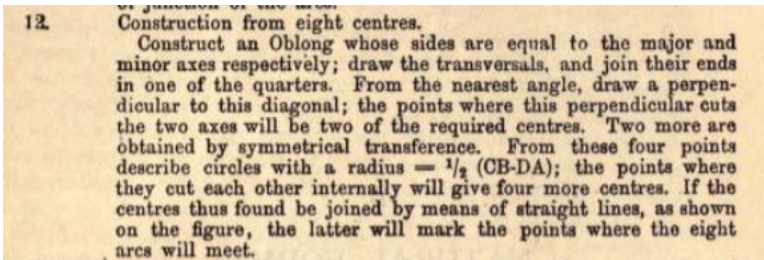


Fig. 21: Description of the construction from the book [6]

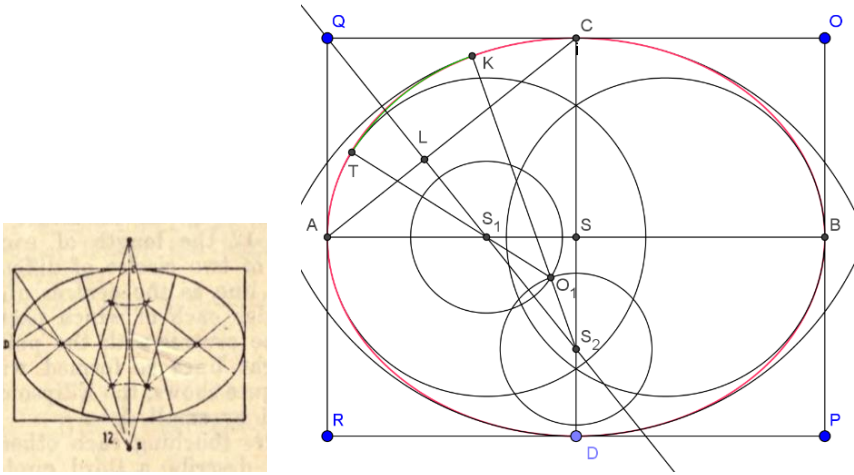


Fig. 22: Left: Meyer's construction in the book [6], Right: Meyer's construction (green colour), ellipse (red colour)

5 Conclusion

The comparative study of Fialkowski's *Zeichnende Geometrie* and Meyer's *Handbuch der Ornamentik* demonstrates that historical geometric constructions are far more than technical exercises — they represent a bridge between mathematical precision, artistic creativity, and cultural heritage. Both authors used ellipses and ovals not only as abstract curves but as instruments for understanding spatial harmony and proportion.

Fialkowski's systematically organised methods, focused on accuracy and pedagogical clarity, provide a foundation for modern approaches to teaching descriptive geometry. His emphasis on visualisation and constructive reasoning aligns with today's goals of fostering spatial imagination and problem-solving skills through dynamic geometry software such as GeoGebra.

Meyer's contribution, rooted in the synthesis of geometry and ornamentation, shows that mathematical forms have enduring aesthetic and functional value. His geometrically structured design system can inspire contemporary STEAM education, bridging art, mathematics, and digital design. The study of ellipses and ovals thus becomes not only a mathematical challenge but also an exploration of pattern, structure, and proportion in visual culture.

From an interdisciplinary perspective, these constructions find practical applications in architecture, optics, acoustics, and industrial design — from the reflective properties of ellipses used in telescopes and medical devices, to the

oval ground plans of amphitheatres, galleries, and urban spaces that enhance visibility and acoustics.

Revisiting these 19th-century sources through modern tools demonstrates how historical geometry can be revitalised in the context of contemporary education, digital modelling, and artistic practice. By reconnecting with the intellectual legacy of Meyer and Fialkowski, we reaffirm that geometry is not a closed historical chapter but a living language linking science, art, and technology.

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Správa zo stretnutia matematika s deskriptívnou geometriou

Report from a meeting between a mathematician and descriptive geometry

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Abstract. The lecture consists of reflections on teaching descriptive geometry to prospective teachers of the subject. As my background is in mathematics, I will try to offer a viewpoint of an outsider. I will inform about implementations of the observations into the current transformations of the course at our faculty.

Keywords: teaching of descriptive geometry

Kľúčové slová: výuka deskriptívnej geometrie

1 Úvod

Vedomosti a zručnosti, ktoré sú predmetom deskriptívnej geometrie, sú na trhu práce žiadané. Napriek bezproblémovému uplatneniu počet študentov deskriptívnej geometrie na vysokých školách dlhodobo a vytrvalo klesá.

Problémy s geometriou na školách sa ale začínajú omnoho skôr než pri výuke deskriptívnej geometrie na univerzitách. Od študentov vieme, že geometriu už na strednej škole nemajú príliš v láske, a to aj tí, ktorí majú matematiku inak radi. Ako dôvody okrem problémov s priestorovým videním uvádzajú aj, že geometria je málo algoritmická: nestačí naučiť sa vzorec (ako napríklad pre počítanie koreňov kvadratickej rovnice) či postup výpočtu (ráčanie s percentami). Pri skúmaní problémov v geometrii musia nahliadať na útvary a ich vzťahy a snažiť sa medzi nimi nájsť tie, ktoré ich dovedú k riešeniu.

Okrem toho nás študenti, ktorí sa venujú stredoškolskému zúčastňujúcim sa matematických súťaží, upozornili, že z výsledkov medzinárodnej matematickej olympiády je zjavné, že slovenskí študenti majú oproti ostatným účastníkom výrazne väčšie problémy práve s príkladmi z geometrie. Teda fakt, že absolventi stredných škôl od geometrie bočia, nie je spôsobený len samotnou povahou disciplíny. Niečo nie je v poriadku s výukou geometrie už na slovenských stredných školách.

2 Geometria pre budúcich učiteľov matematiky

Chcela by som sa preto najprv stručne venovať príprave budúcich učiteľov matematiky z geometrie.

Kvôli trendu redukcie odborných predmetov v prospech pedagogických a didaktických ostali z pôvodne piatich semestrov geometrie dnes iba tri, aj z nich iba dva povinné, tretí povinne voliteľný. Pred časom som postupne prevzala výuku celého tohto kurzu geometrie a vďaka tomu som získala vhľad, čo sa v ňom deje. Stručne by som situáciu zhrnula tak, že kurz sa rozhodne nestíhal prispôbovať postupnému kráteniu časovej dotácie. Stručne načrtnem, ako som výuku geometrie nastavila.

2.1 Analytická geometria

Celá analytická geometria je pokrytá jediným semestrom. Našťastie nejakú lineárnu algebru študenti už za sebou majú, takže netreba začínať úplne od nuly. Obsahovo semester pokrýva afinné priestory a ich podpriestory (lineárne variety), vzájomné polohy, metrické úlohy, afinné zobrazenia, vlastné čísla a vektory, samodružné body, priamky a roviny (posledné len na úrovni definície), na záver zhodnosti roviny a ortogonálne matice.

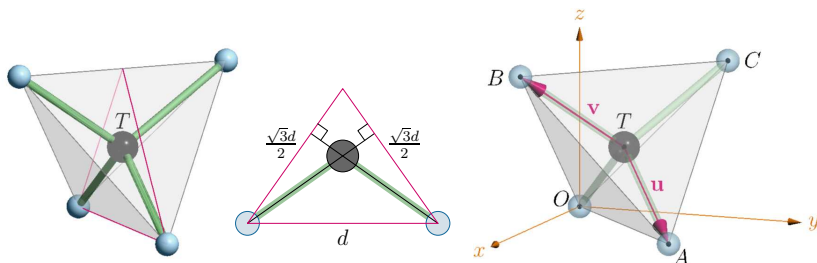
Kvôli krátkosti času sa obmedzujeme na výpočty v dvoj- a trojrozmernom priestore, viacrozmerné situácie sa uvedú len pre ilustráciu. Cieľom je, aby sa študenti naučili používať metódy analytickej geometrie, transformovať geometrický problém do súradníc a nájdením riešenia rovníc nájsť odpoveď na otázku.

2.2 Axiómy rovinatej geometrie

Druhý semester je venovaný konštrukčnej geometrii.

Pri jeho navrhovaní som zvažovala, či sa zamerať na axiomatickú výstavbu planimetrie, alebo sa so znalosťami stredoškolskej geometrie venovať štúdiu zaujímavejších a pokročilejších tém ako Feuerbachova kružnica, kružnicová inverzia, Apollóniove úlohy a podobne, keďže obidvoje sa rozumne stihnúť nedalo. Po diskusiách s niekoľkými učiteľmi na stredných školách a po zisťovaní, ako vyzerá geometria na stredných školách v zahraničí, som sa rozhodla pre axiomatiku, pričom sa jej študenti venujú aj na cvičeniach. Cieľom je, aby sa študenti zdokonalili v dokazovaní matematických tvrdení, aby sa naučili korektne argumentovať a svoje argumenty zapisovať.

Podľa niektorých vyjadrení študentov sa zrejme darí vytýčený cieľ aspoň ako-tak dosahovať: viacerí študenti s nadšením konštatujú, ako *sami* našli dôkaz nejakého tvrdenia. Niektorí tiež nahliadli, že síce samotný obsah predmetu vo svojej učiteľskej praxi nepotrebujú, no schopnosť dokazovať, ktorú získali, oceňujú. Verím teda, že rozhodnutie zamerať sa na axiomatiku na úkor pokročilejších tém planimetrie bolo v danej situácii dobré.



Obr. 1: Hľadanie uhla väzieb molekuly metánu.

2.3 Stereometria

Aby som pokračovala v tradícii, že každý semester je zameraný na nejakú zručnosť, ktorú zdokonaľuje, celkom prirodzene tu vyvstalo priestorové videnie. Naozaj ho považujem za zručnosť, ktorá sa pestuje a zdokonaľuje tréňovaním. Určite, že niekomu sa darí viac a inému menej, ale tak je to predsa s každou zručnosťou.

Ak by som mala stručne popísať obsah tohto predmetu, snažila som sa nájsť také témy, ktoré študenti vidia ako relevantné pre svoju skúsenosť. Tak grupy symetrií majú veľmi pozitívny ohlas u tých študentov, ktorí riešili matematické olympiády a nebočia od abstraktných pojmov. Tvary a geometria chemických molekúl (stereochemia) oslovujú budúcich učiteľov chémie. Konštrukcia scény renesančných obrazov pomocou lineárnej perspektívy snáď je zaujímavou pre milovníkov výtvarného umenia. Budúci učitelia geografie vždy oceňujú geometriu sféry a vlastnosti kartografických zobrazení.

Uvedomujem si, že pestosť a príťažlivosť predmetu ide trochu na úkor hĺbky a dôslednosti: než by sa naozaj naučili niečo solidne počítať, sa študenti skôr dozvedia, kde všade sa geometria nachádza, a získajú tušenie, že existujú metódy (všetne tých analytických!), ktorými sa tieto problémy dajú vyriešiť. A to bol v podstate môj cieľ: chcem, aby si podľa možnosti každý študent našiel v stereometrii niečo, čo mu je blízke, a odniesol si tak z tejto školy ku geometrii pozitívny vzťah, lebo len tak ho môže posunúť ďalej svojim žiakom.

3 Deskriptívna geometria

Predchádzajúca kapitola myslím dostatočne ilustrovala, ako som pristupovala k prepracovávaniu kurzu geometrie: najprv sa pozrieť na materiál, ktorý je kurzom pokrytý, a následne sa zamyslieť nielen nad obsahom vedomostí, ktoré má kurz priniesť, ale aj nad zručnosťami, ktoré si kladie za cieľ rozvíjať.

3.1 Úvodné úvahy po pohľade na kurz

Jadrom disciplíny, ktorá sa nazýva „deskriptívna geometria“, sa javia byť zobrazovacie metódy. Na našej škole sú v príprave budúcich učiteľov deskriptívnej geometrie zobrazovacím metódam venované štyri semestre s výdatnou časovou dotáciou štyroch, prípadne až piatich hodín za semester.

Prvý pohľad do obsahu predmetu ma veľmi rýchlo priviedol k presvedčeniu, že deskriptívna geometria je terminologicky preťažená disciplína a navyše ju jej terminologická záťaž značne izoluje od ostatnej matematiky. Mohutný terminologický aparát tvorí stenu, ktorá odrádza prípadných záujemcov z príbuzných odborov. Ako príklad by som spomenula „súmiestne a nesúmiestne rovinné polia“. Rozumiem tomu tak, že pojmom „rovinné pole“ (namiesto jednoduchého a známeho „rovina“) sa zrejme nemyslí len množina bodov roviny, ale aj všetky objekty roviny, s ktorými pracujeme (priamky, trojuholníky, ...). Avšak je otázne, či kvôli tomuto faktu zavádzať nový pojem. Ja tiež pod „rovinou“ chápem nielen jej body, ale celú bodovo-vektorovú štruktúru, ktorú v nej viem v záujme svojej pohodlnej práce vybudovať. Dokonca je v matematike zvykom reálnu *afinnú* rovinu označovať $\mathbb{R}^2$, teda presne tak isto ako vektorový priestor nad reálnymi číslami. Podobne pojem „homológia“ ma chvíľku miatol, kým som zistila, že neodkazuje na postupnosť ábelovských grúp priradených komplexu, ale na osovú kolineáciu roviny (ktorá má v deskriptívnej geometrii ešte aspoň ďalšie dve mená: stredová kolineácia a perspektívna kolineácia).

Pripúšťam, že ide o nedorozumenia, ktoré sa rýchlo vyjasnia, avšak ak je takýchto zakopnutí pri vstupe k deskriptívnej geometrii veľa, potenciálneho adepta to skôr odradí namiesto toho, aby sa cítil vítaný a povzbudzovaný vstúpiť.

Viem o ďalšej matematickej disciplíne, ktorá má mohutný a odstrašujúci jazyk: algebraická geometria. Adept potrebuje kvalitné doktorandské štúdium, aby začal jej jazyku aspoň trochu rozumieť. Avšak algebraická geometria si môže tento luxus dovoliť, sú v nej totiž výzvy a zaujímavé otvorené problémy, ktorými k sebe ako magnet priťahuje impresívne matematické mozgy.

Deskriptívna geometria však paletu otvorených problémov neponúka. Odráža sa to aj na fakte, že deskriptívnu geometriu (na rozdiel od fyziky, informatiky, chémie, biológie, ...) nie je možné študovať ako vedeckú disciplínu. Naozaj, učiteľské štúdium deskriptívnej geometrie je to najobľahšie štúdium tohto odboru, aké univerzity ponúkajú. Takže ako sa odbor (napríklad matematika) pestuje a posúva dopredu so študentami odborného štúdia, tak sa deskriptívna geometria pestuje so študentami učiteľstva tohto predmetu.

Keď sa ale pozriem bližšie, čo na našej škole znamená „pestovať deskriptívnu geometriu“, pozorujem, že aj tu disciplína výrazne trpí tým, že je uzavretou vednou disciplínou: stretávam sa s pomerne komplikovanými definíciami elementárnych pojmov (vnútorný či vonkajší bod plochy sa opiera o mohutný terminologický aparát, dotýčnica kužeľosečky sa pre každý typ definuje zvlášť pomocou jej špeciálnych vlastností v každom z prípadov) a následne sa vyslovia a vyčerpávajúco dokazujú ich často pomerne evidentné vlastnosti. Akoby sa disciplína po tom, ako prestala expandovať a generovať nové poznatky, obrátila dovnútra seba samej a svoju aktivitu nasmerovala na dôkladné dokazovanie v podstate až na úroveň axióm. Zdá sa mi nepatričné, aby sa s takýmto systematickým a únavným dokazovaním stretali študenti v prvom ročníku.

Na druhej strane, v opačnom extréme, by som spomenula môj obľúbený príklad: konštrukcia obrazu kružnice v osovej afinity v sebe de facto ukrýva krásny geometrický dôkaz o singulárnom rozklade matice pre dvojrozmerný prípad, čo je zaujímavé tvrdenie v lineárnej algebre. Deskriptívna geometria ale v zameraní sa na drobné detaily akoby stratila takýto globálnejší náhľad a prezentuje tento fakt len ako návod na konštrukciu. Tým chcem ilustrovať, že je síce pravda, že deskriptívna geometria je do istej miery viac remeslom než vednou disciplínou, niekedy je však týmto smerom zatísnutá hlbšie, než je primerané.

3.2 Ako učiť zobrazovacie metódy inak?

Bohužiaľ v tomto smere som len na začiatku a veľa odpovedí nemám. Mám ale mnoho otázok a pozorovaní, ktoré sa mi zdajú byť vhodné do diskusie a som veľmi vďačná, že som dostala možnosť ich na tomto fóre predniesť a hneď som získala aj mnoho postrehov, názorov a skúseností.

Riešiť teda chcem dve stránky:

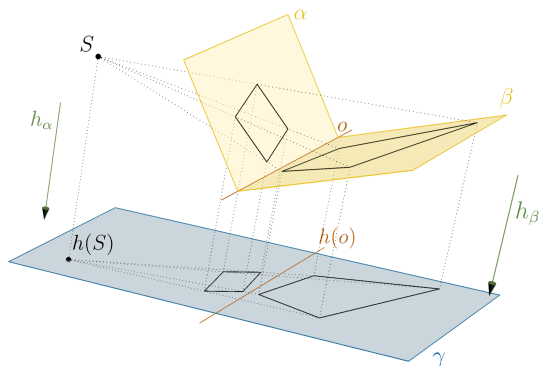
- obsah, ktorý kurz prináša,
- zručnosť, ktorú má kurz trénovať.

3.3 Obsah úvodu kurzu

Isté zmeny v štruktúre materiálu som už urobila, avšak domnievam sa, že to boli len prvé a pomerne jednoduché úpravy.

Pomedzi zobrazovacie metódy sa študenti zoznamovali s osovými afinitami a kolineáciami, no vnímané boli (použijem rečnícku hyperbolu) hlavne ako „niečo, s čím sa pracuje takto“ a strácal sa matematický význam týchto tried transformácií. To mi prišla byť škoda, preto prvá úprava spočívala v tom, že študenti sa v úvode systematickejšie zoznámia s transformáciami roviny.

Najprv preštudujeme zhodnosti roviny. Ukážeme, ako osová súmernosť generuje celú grupu izometrií, a je príležitosť stretnúť sa aj s posunutou súmernosťou. Pridaním rovnolehlosti grupu zväčšíme na podobnosti.



Obr. 2: Osová kolíneácia je rovnobežným priemetom stredového premietania medzi rovinami.

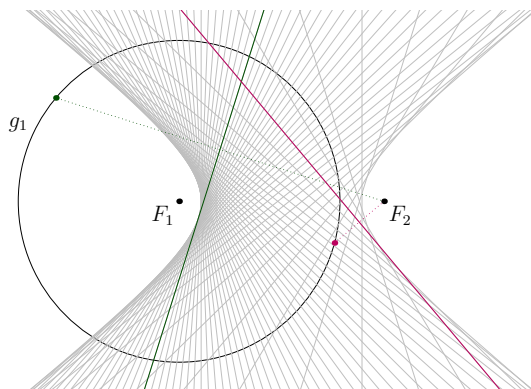
Následne sa pomerne dôkladne venujeme afinitám roviny. Pri tomto prístupe sa osovú afinitu ukázu ako významné nielen pre deskriptívnu geometriu ale aj pre jej teoretický základ ako generátory celej grupy afín. Za dôležité pre neskoršie konštrukcie považujem chápanie osovej afinity ako rovnobežného priemetu rovnobežného premietania medzi dvoma rovinami. Značnú pozornosť venujeme elipse. Ukážeme, že obrazom kružnice v *akejkol'vek* afinite je elipsa (spomínaný singulárny rozklad matice).

Nakoniec pristúpime ku kolíneáciám. Tu sa už zameriavame špeciálne na osovú kolíneáciu, všeobecnejšia teória je pokrytá predmetom Projekatívna geometria. Podobne ako pri afinitách, dôležitým je nahliadanie na osovú kolíneáciu ako na rovnobežný priemet stredového premietania medzi rovinami.

Prirodzene tak prechádzame k teórii kužeľosečiek. Táto téma sa mi zdala v tomto kurze vzhľadom na jej dôležitosť značne zanedbaná.

U všetkých typov kužeľosečiek sa študenti oboznámia s metrickou definíciou kužeľosečky, odvodíme rovnicu v kanonickom tvare – kanonický tvar je v tejto fáze bohaté postačujúci. Pomocou QD-viet sa študenti presvedčia, že kužeľosečky definované metricky sú skutočne rezmí kužeľovej plochy. A asi za najdôležitejšie považujem chápanie kužeľosečiek ako obrazov kružnice v kolíneácii. Toto vnímanie je totiž bezkonkurenčne užitočné pre zavedenie základných pojmov ako vnútorný a vonkajší bod, dotyčnica a pre zdôvodnenie ich základných vlastností.

Tu máme za sebou takmer jeden a pol semestra. Náročná časť transformácie kurzu je ale stále pred nami. Až teraz prichádzajú na rad samotné zobrazovacie metódy. Aby som sa s témou vedela zmysluplne popasovať, potrebujem stále zvažovať, o čom zobrazovacie metódy vlastne sú.



Obr. 3: Hyperbola ako obálka svojich dotyčníc.

3.4 Význam zobrazovacích metód

Predpokladám, že zobrazovacie metódy sa vyvinuli, pretože bolo potrebné:

- reprezentovať a uložiť/archivovať modely priestorových objektov (zastaraný, dnes už neaktuálny dôvod),
- komunikovať dizajn modelu (stále aktuálna motivácia).

Evidentnou zručnosťou, ktorá sa v minulosti pestovala zobrazovacími metódami, bolo klasické rysovanie kružidlom a pravítkom. Pochopiteľne je to zručnosť, ktorú, chceme či nechceme, musíme ak nie nahradiť, tak aspoň doplniť používaním CAD-systémov.

Prirodzene sa môžeme pýtať, či vôbec nielen rysovanie, ale dokonca klasické zobrazovacie metódy potrebujeme. Nebudeme objekty pomocou počítačov modelovať priamo v troch rozmeroch? A vtedy aj ak pre akékoľvek dôvody potrebujeme jeho rovinné zobrazenie, ide o jasné zadanie pre počítač, v ktorom je objekt vymodelovaný.

Avšak ak sa prizriem pozornejšie, vidím, že „zastarané umenie“ rysovania zobrazení modelov je veľmi úzko prepojené s ďalšími zručnosťami, ktoré, predpokladám, považujeme za dôležité aj dnes, konkrétne:

- znázornenie objektu pomocou zobrazovacej metódy pomáha mentálne uchopiť trojrozmernú situáciu z jej rovinných obrazov,
- riešenie incidenčných a metrických úloh posilňuje schopnosť zručne manipulovať s priestorovými útvarmi.

Ide tak o excelentný „learning by doing“ prístup pre 3D kompetenciu.

3.5 Zobrazovacie metódy v deskriptívnej geometrii

Zobrazovacie metódy sú z pohľadu matematiky vybavené definíciou a základnými vlastnosťami rovnobežného a stredového premietania. V pro-

jektívnom priestore dokonca aj tieto dva typy premietania splynú do jedného. Mnohé následné úsilie deskriptívnej geometrie sa dotýka vzťahu súradnicových sústav priestoru na jednej strane a priemetne na druhej.

Obsahom kurzu zobrazovacích metód u nás sú [3]

- kótované premietanie,
- Mongeova projekcia,
- kolmá axonometria,
- šikmé premietanie,¹
- kosouhlá axonometria,
- stredové premietanie.

Každá z metód sa študuje ako samostatný celok. V každej sa najprv rieši obraz základných útvarov, potom incidenčné a následne metrické úlohy.

Stredové premietanie z nasledujúcich úvah vynechám a budem sa venovať rovnobežným premietaniam. Z nich špeciálnu pozíciu by som prisúdila Mongeovej projekcii. Chcela by som tu spomenúť, že keď som sa študentov učiteľstva matematiky po absolvovaní kurzu geometrie pýtala, čo im pomohlo v precvičovaní si priestorového videnia, odpovedali mi, že užitočné bolo konštruovanie rezov telies, ale najviac im pomohla práca s Mongeovou projekciou. Už toto samé je pre mňa dostatočný dôvod na to, aby sa Mongeovej projekcii venovala zvláštna pozornosť, ani nehovoriac o tom, že pre technické účely ide o zvlášť dôležité zobrazenie.

Kótované premietanie sa mi javí ako modifikácia Mongeovej projekcie, ktorá je vhodná pre viaceré aplikácie. Avšak venovať sa jej zvlášť a s rovnakou precíznosťou vidím ako nevhodné nakladanie s časom, keďže všetky postupy, s ktorými sa študenti v súvislosti s kótovaným premietaním stretnú, sú de facto len kópiami metód Mongeovej projekcie.

Ostatné metódy rovnobežného premietania by som v záujme lepšieho nadhľadu predstavovala súhrnne ako „axonometriu“. Pohlkeho veta sa vzťahuje na všetky z nich, ba dokonca zahŕňa aj situácie, ktoré šikmé premietanie, kosouhlá ani kolmá axonometria nepokrývajú. Navyše incidenčné úlohy sa riešia vo všetkých týchto zobrazeniach rovnako. Preto až po zvládnutí incidenčných problémov by som predstavila šikmé premietanie a kolmú axonometriu ako špeciálne prípady, ktoré prinášajú každá svoje efektívne metódy pre riešenie metrických úloh – rozumiem tomu

¹Mimochodom, zas oproti štandardnej terminológii v matematike dochádza k posunu. Napríklad „šikmé premietanie“ v deskriptívnej geometrii neznamená len to, že smer rovnobežného premietania nie je kolmý na priemetňu, ale aj fakt, že súradnicové osi priemetne splyvajú s dvoma zo súradnicových osí priestoru. Pojem „izometria“ v deskriptívnej geometrii sa nevzťahuje na vlastnosť zobrazenia, ale vypovedá niečo o vzťahu premietania a súradnicovej sústavy priestoru.

tak, že presne toto je dôvodom pre aktuálny výrazný dôraz na rozlíšenie týchto metód.

Omnoho zásadnejšia otázka, ktorú si však kladiem, je: potrebujeme vôbec riešiť metrické úlohy priamo povedzme v kosohľej axonometrii? Keď si zalistujem aj v starších učebniciach deskriptívnej geometrie [4], tak vidím, že už pred desiatkami rokov boli študenti povzbudzovaní k tomu, aby vzdialenosť bodov zobrazených v Mongeovej projekcii zisťovali radšej výpočtom než konštrukčne! Prečo teda dnes tak veľmi trváme na konštrukčných metódach pri riešení metrických úloh?

Navrhla by som namiesto elementárnych úloh s komplikovanými riešeniami (napríklad vzdialenosť bodov zobrazených v spomínanej kosohľej axonometrii) viac času venovať rovnobežnému premietaniu všeobecne. Študenti nech dôkladnejšie porozumejú vzťahu medzi súradnicovými sústavami priestoru a priemetne. Nech vedia manipulovať s premietaním viac spôsobom, ako sa to robí v počítačovej grafike, a nie sú limitovaní tým, že vedia pracovať až vtedy, keď majú zadané priemety súradnicových osí priestoru. Cieľom je naozaj dať študentom čo najväčší nadhľad, aby boli aj v budúcnosti čo najflexibilnejší.

Týmto by študenti mohli byť pripravení na to, aby vedeli pracovať s axonometriou všeobecne. Šikmé premietanie vedú potom nahliadnuť ako jeden z prípadov axonometrie. A v neposlednom rade kolmá axonometria by získala výrazné postavenie vďaka svojím z viacerých aspektov výhodným vlastnostiam.

3.6 Zručnosť tréňovaná zobrazovacími metódami

Prejdem naspäť k zručnostiam, ktoré chceme u študentov posilniť. Ako sa postaviť k rysovaniu s kružidlom a pravítkom? Je to zastaralá zručnosť, ktorú treba ju nahradiť modernými nástrojmi? Každý z protichodných postojov má svoje opodstatnené dôvody:

- prečo by študenti mali rysovať:
 - (i) vďaka dôkladne precvičenému rysovaniu si študenti trúfnu pred triedou na tabuľu črtať (budúci učitelia deskriptívnej geometrie),
 - (ii) aby študenti zvládli prácu s CAD systémami, musia toho mať najprv dosť odrysovaného (budúci inžinieri),
 - (iii) rysovanie predstavuje fyzický kontakt s predmetom štúdia a je možné, že tento kontakt je pre človeka dôležitý.
- prečo by sme nemali na rysovaní trvať:
 - (iv) máme tu éru CAD a dokonca aj generatívnej AI; v tejto situácii je dôležité rysom *rozumieť* a nie ich ručne vytvárať,

- (v) je pochopiteľné, že kultúra rysovania u študentov upadá: veď prečo by študenti mali rysovať, keď ani ich učitelia už toľko nerysujú?

K bodu (v): keď tvoríme učebné materiály či rozdávame študentom pracovné listy, zriedka obrázky ručne narysujeme, spravidla siahame po digitálnych nástrojoch. Študenti (nielen deskriptívnej geometrie) tiež obrázky pre svoje záverečné práce nerysujú, ale ich vytvárajú elektronicky, zvyčajne (takmer výlučne) pomocou GeoGebry.

Kým GeoGebra má svoje neodškriepiteľné výhody (názornosť, dynamické konštrukcie), pri náročnejších konštrukciách, keď cieľom je presne zobrazíť modelovaný objekt, narážame na jej hranice (nepresnosť, nestabilita, zdĺhavosť). Ja osobne na druhej strane svoje obrázky vytváram pomocou asymptote. Študentov mám však problém na asymptote nahovoriť, pre jeho použitie totiž potrebujú dve netriviálne zručnosti:

- vyjadrovať sa jazykom analytickej geometrie,
- programovať.

Analytickú geometriu vrámci štúdia zvládnu na dostatočnej úrovni, s programovaním sa viacerí počas štúdia nestretnú. Iní sa aj stretnú, no rýchlo to zabudnú a neskôr od programovania bočia. Potrebujeme túto zručnosť promptne podchytiť a ďalej rozvíjať.

K bodu (ii): bez ohľadu na to, čo si myslíme na tému „rysovať alebo nerysovať“, na štúdiu učiteľstva deskriptívnej geometrie u nás je práca s CAD-systémami výrazne zanedbaná. Neskôr v štúdiu sa síce študenti na niektorých predmetoch s týmito systémami stretnú, no je to príliš neskoro. Myslím si, že by sa tieto systémy mali využívať hneď, ako sa študenti začnú so zobrazovacími metódami zoznamovať. Práve CAD-systémy z môjho pohľadu v sebe spájajú aj názornosť GeoGebry (jednoduchý obrázok sa dá proste „naklikať“), aj presnosť, akú dosiahneme pri programovaní. Kurz zobrazovacích metód na začiatku štúdia je vynikajúcou príležitosťou zoznámiť sa s jednoduchšími, 2D CAD systémami. Pre kvalitné zvládnutie systému potrebujeme následne učiť študentov v CAD-e programovať skripty a tým ich viesť k pozitívnemu vzťahu k programovaniu.

K bodu (iv): Vidím to ako veľkú otázku pre didaktiku deskriptívnej geometrie: ako učiť študentov rozumieť rysom? Myslím, že to nebude celkom možné bez toho, aby študenti aspoň niečo nerysovali ručne. Treba zrejme nájsť rozumnú hranicu: aké množstvo ručného rysovania je potrebné a kedy je to už len neužitočná a pracná robota?

V tejto súvislosti by sme mohli diskutovať už o tom, či aj na stredných školách rysovanie v záujme modernizácie nenahradiť prácou napríklad v GeoGebre, ktorá je pre tieto školy určená. Čo je vlastne účelom konštrukčných úloh na stredných školách? Je cieľom

- trénovať prácu s pravítkom a kružidlom? V tom prípade najpodstatnejšou časťou riešenia je vykonanie konštrukcie.
- učiť sa hľadať riešenie problému? Vtedy by najpodstatnejšou časťou riešenia bol rozbor konštrukčnej úlohy.

Priznám sa, že sama sa zdráham namiesto kružidla a pravítka dať žiakom GeoGebru. Aby som samu seba presvedčila k modernejšiemu prístupu, odhodlala som sa viesť na túto tému bakalársku prácu [1]. Moje pochybnosti v tomto smere bohužiaľ nerozptýlila.

Tú istú otázku ako pre školské konštrukčné úlohy si môžeme položiť aj pre konštrukčné úlohy v stereometrii, s ktorými sa stretávame v deskriptívnej geometrii. Tiež vidím dva ciele, na ktoré tieto úlohy miera:

- cvičiť sa v riešení problémov,
- posilňovať svoju 3D kompetenciu.

Obidva ciele sú dôležité. Mohli by sme skúsiť diskutovať o tom, ako najlepšie naplniť cieľ trénovať priestorové videnie. Aj priamo na konferencii som sa nie len raz stretla s vyjadrením, že úlohy v stereometrii treba riešiť v Mongeovej projekcii, a s týmto názorom súhlasím: riešiteľovi to dá v tomto smere podstatne viac než keď zaňho urobí veľkú časť práce 3D modelovací software.

To však neznamená, že študenta nebudeme oboznamovať s tým, aké metódy modelovací software používa!

4 Záver

Deskriptívnu geometriu ako outsider vidím ako disciplínu, ktorá sa kvôli svojej vedeckej uzavretosti a tiež izolovanosti začala akoby rútiť sama do seba. Ak sa chceme vyhnúť tomu, aby sa týmto zrútila aj do minulosti a zabudnutia, musí sa z môjho pohľadu otvoriť ďalším disciplinám. Sami vidíme, že by bolo prínosné, keby sa študenti stali digitálne gramotnejšími, otvorenejšie sa stavali k programovaniu a nielen formálne ale reálne používali analytickú geometriu. Mali by mať väčší nadhľad nad svojím predmetom a ten im vedľa poskytnúť iné geometrické disciplíny.

Uvažujem, či by bolo možné, aby deskriptívna geometria samu seba poňala ako multidisciplinárnu oblasť. Ako napríklad kognitívna veda, ktorá je dialógom veľmi pestrej zbierky disciplín (filozofia, psychológia, neuroveda, informatika a umelá inteligencia). Prípadne ako sme aj na tejto konferencii počuli, že architektúra v sebe kombinuje geometriu, výtvarné umenie, psychológiu, sociológiu aj ekonómiu [2] a presne to robí architektúru živou a príťažlivou. Vedela by deskriptívna geometria nájsť a aktívne vyživovať kontakty s inými vedami?

Obávam sa, že ak deskriptívna geometria ostane uzavretou a izolovanou, bude sa musieť zmieriť s tým, že dlhodobo neprežije.

Na záver by som uviedla slová, ktoré boli síce vyslovené v úplne inom kontexte, ale neviem si predstaviť výstižnejšiu charakterizáciu aktuálneho stavu deskriptívnej geometrie:

„Uzavretý systém speje k entropii. Len otvorený systém môže rásť a vyvíjať sa.“ (Ilia Delio)

Pod'akovanie

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Aesthetic curve families in computer - aided design

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Abstract. The beauty or "fairness" of geometric models is of central importance in computer-aided design, especially in areas such as the design of car body panels. In the usual workflow, models are created based on 2D sketches containing feature curves of the object. The quality of these curves exert consequently a large influence on the overall shape, so we need them to be "aesthetic".

But what do we mean by aesthetic curves? In this talk, we will explore this question by revisiting classical curves generally regarded as beautiful. This will lead us to families of aesthetic curve representations, with a deeper look into so-called log-aesthetic curves and their variations. We will also examine their intriguing connection to Archimedean spirals, offering insights into possible applications

Editor's comment

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www.ssgg.sk/G/Abstrakty/indexa.html

Current challenges and perspectives of descriptive geometry

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Abstract. This contribution is a response to the discussion on the current state of descriptive geometry in the Czech and Slovak Republics. We will draft the current problems that descriptive geometry encounters in the field of education, applications, and research. We will identify the original motivations as fixed elements accompanying the subject since its foundation. The aim of the lecture is to propose projections of these motivations to the current time into two interrelated projection planes: descriptive geometry as a part of mathematics and as an application in other subjects of not only technical but also general education.

Keywords: descriptive geometry, education, curriculum

1 Introduction

“Enough! Just enough of this cold,” Étienne muttered to himself as he finally approached his drafting desk in Mézières. The pain in his frozen, cramping fingers was inhuman as he spread out the nearly finished plan of the Cherbourg fortification on his drafting table. “Just a few more lines to keep the Godons in their soup,” he urged himself on, while the heat in his forehead spread across his chilled face.

“Really? Not again!” exhaled Maria desperately when her home Wi-Fi was interrupted by provider maintenance. Now I’ll spend half the day blueprinting these matrices myself, she muttered, pressing the button on the espresso machine. “Okay, okay, focus — this is important,” she said, calming herself as she sat back down at her laptop with a fresh coffee. She began to type, self-confident and a little impressed by her progress, determined to present the software refinements for graphical cancer diagnostics at the consortium tomorrow.

Both Étienne — a military engineer working on the project in 1782 — and Maria — a postdoc at a medical visualization company in 2025 — work on important tasks involving geometric visualization. They both need to visually imagine the results of their work, understand geometric properties and transformations, and use appropriate representations to present their projects. However, their tools and representations differ significantly.

The parallel above illustrates the changing perspective on descriptive geometry. On one hand, the competencies that must be developed remain invariant; on the other hand, the specific content, tools, and applications change dramatically over time. Yet, the teaching of school geometry and descriptive geometry has remained largely untouched—from elementary school to the university level. Consequently, descriptive geometry with obsolete content becomes obsolete itself. We will further develop these thoughts and attempt to outline some conclusions in the following lines.

2 Initial motivations

As descriptive geometry spread in the 19th century, it naturally absorbed various motivations, aims, and methods. We hypothesize that these shifts, when not re-evaluated, led to the obsolescence and disappearance of descriptive geometry in most countries. Hence, for our reflection, it is essential to revisit the initial conditions and goals stated by Gaspard Monge [?]. We encourage the reader to examine Monge's motivations and first steps in his *Programme* through the eyes of a 21st-century student, teacher, engineer, or researcher.

- *to direct national education toward the knowledge of objects that demand exactitude —something which has been entirely neglected until today—and to accustom the hands of our artisans to the use of instruments of every kind which serve to bring precision into works and to measure its various degrees. ...*
- *to make popular the knowledge of a great number of natural phenomena, indispensable to the progress of industry ...*
- *to spread among our artisans the knowledge of artistic processes, and of machines whose object is either to reduce manual labour, or to give the results of work greater uniformity and greater precision ...*

Later in the text, Monge stated the objectives of descriptive geometry as follows:

- *to represent with exactness, upon drawings which have only two dimensions, objects which have three, and which are susceptible of rigorous definition. From this point of view, it is a language necessary to the man of genius who conceives a project, to those who must direct its execution, and finally to the artisans who must themselves execute its different parts.*
- *to deduce, from the exact description of bodies, everything that necessarily follows from their forms and their relative positions. In this sense, it is a means of seeking truth: it offers perpetual examples of the passage from the known to the unknown; and because it is always applied to objects susceptible of the greatest evidence, it*

must be included in the plan of national education. It is not only suited to exercising the intellectual faculties of a great people, and thereby contributing to the improvement of the human species, but it is also indispensable to all workers whose aim is to give bodies certain determined forms.

From the previous lines, we can identify three interconnected essences of descriptive geometry: the theoretical framework and language based on the exact mathematical representation of objects and methods; the use of suitable available tools to obtain the best results; and the application of constructive methods and visualizations to contemporary problems.

It is important to note that Monge did not intend to separate synthetic and analytic geometry; on the contrary, he emphasized their interrelation: *It is not without purpose that we compare descriptive geometry to algebra here; these two sciences have the most intimate relations. There is no construction in descriptive geometry that cannot be translated into analysis; and when the questions do not involve more than three unknowns, each analytic operation may be regarded as the written transcription of a spectacle in geometry. It would be desirable that these two sciences be cultivated together: descriptive geometry would bring to the most complicated analytic operations the clarity and evidentness which are its hallmark, and analysis, in turn, would bring to geometry the generality which is its own.*

Although the subject of descriptive geometry is built on solid aims and motivations, it is losing its struggle with time. A good overview of the decline of descriptive geometry in various countries is given in [?]. We will briefly describe the two most common ways in which descriptive geometry has disappeared.

First, descriptive geometry became purely mathematical and theoretical. One of the positive outcomes of this shift was its evolution into projective geometry. Let us recall the words of Cayley [?, p. 90], where “descriptive” is used in the sense of “projective”: *Metrical geometry is thus a part of descriptive geometry, and descriptive geometry is all geometry.* Consequently, projective geometry began to be treated through algebraic representations, disconnecting itself from the cumbersome language of descriptive geometry. The strength of analytic representation overruled the intuitiveness of visual constructive methods. The language of descriptive geometry became obsolete and useless for further mathematical exploration.

On the other hand, descriptive geometry became a purely technical subject focused on applications. In other words, it transformed into technical drawing. However, such a subject can remain relevant only if its content adapts to new tools and applications. In most cases, the curricu-

lum of descriptive geometry failed to do so, continuing to rely on outdated examples, tools, and methods.

Returning to Monge's initial ideas, we conclude that descriptive geometry can be successful only if it keeps pace with current applications and tools while employing appropriate mathematical representations.

3 Reflection on the current state of descriptive geometry education in the Czech Republic

Descriptive geometry in the Czech Republic is built upon a strong tradition developed primarily by technical universities. For many years, it combined drafting precision with a solid theoretical framework derived from mathematics and projective geometry. However, it did not adapt to the technological progress of the late 20th century. The content and methods of descriptive geometry at high schools have changed very little since its beginnings in the second half of the 19th century. The main focus remains on methods of projection and the goal of creating precise hand drawings. At some point, descriptive geometry became a subject relying exclusively on synthetic constructions while avoiding analytic representations. This approach is illustrated by Sobotka [?]: *Deduction in descriptive geometry is based mainly on methods of projection and in a purely geometrical way . . .*. The emphasis on precision also became an educational goal—sometimes taken to extremes, as seen in [?]: *Descriptive geometry leads pupils by its nature to precision, clarity, order, and discipline.*

We take the liberty of outlining several critical points regarding the regression of descriptive geometry education:

- The content based purely on methods of visualization is obsolete for both mathematical and technical purposes. If visualization methods once served as the primary medium for producing precise representations in practice, they have now been surpassed by 3-D computer-aided modeling and continuous technological development. Therefore, the curriculum must be adapted to reflect this shift. The theoretical aspects of projection remain essential for understanding, but their applications vary depending on the tools used.
- The applications used in descriptive geometry must reflect current tools and topics and be regularly re-evaluated.
- The emphasis on purely synthetic methods and manual drawings disconnects the subject from real-world applications. Instead, analytic and synthetic representations should complement one another, with attention given to their appropriate and effective implementation.

- Descriptive geometry is not an exclusively technical subject. While technical universities employ its instrumental side for engineering applications, the competencies developed through descriptive geometry are highly relevant to many other fields— from mathematics, physics, and computer science to art, medicine, etc. . Therefore, secondary school education in descriptive geometry should be more broadly oriented and not viewed merely as preparation for technical universities. This consideration also applies to teacher training, where the number of graduates is extremely low. Clearly, if most high school graduates of descriptive geometry proceed directly to technical universities, who will study to become teachers of descriptive geometry?

4 Conclusion

The muse of descriptive geometry (Figure ??) is sinking deeply into the mud. Having been preserved for decades, she has become fragile. We have attempted to show that she is worth saving and the aims and competencies developed through descriptive geometry are universally valuable—regardless of time.



Fig. 1: The muse of descriptive geometry.
Source: AI generated in ChatGPT.

An example of a successful transition can be found in the Austrian curriculum. It reflects not only technological advancement in its content but also the reformed educational system based on competence-oriented teaching [?]. Of course, such a document would remain merely a piece of paper for teachers if it were not supported by other interconnected activities, including workshops, journals [?], and textbooks [?]. Let this serve as inspiration for our further steps.

It is important to note that the discussion about the innovation of descriptive geometry in the Czech Republic has already begun. There are specific tasks to be undertaken in the near future. First, we must re-evaluate the content—from the perspective of technical universities and their modern applications and tools; from the perspective of teacher training, to ensure it develops the necessary competencies; and from the perspective of high school teaching, both at general gymnasiums and specialized technical schools, to make the subject useful, modern, and attractive for contemporary needs. We must rebuild communication channels among all these levels, especially ensuring the transfer of current applications and modern tools from universities to high schools. Curricular changes must also be supported by a solid didactic framework.

Hand in hand— theory and application, visual and symbolic, hand-drafted and computer-modeled—descriptive geometry encompasses it all. Therefore, we hope that this is not the conclusion of our contribution, but rather a continuation of our work.

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CONTRIBUTED TALKS

Po stopách Feuerbachovej hyperboly: historický prehľad a geometrické vlastnosti

Feuerbach hyperbole: historical overview and geometric properties

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Abstract. In this article, we present a comprehensive view of the Feuerbach hyperbola through a historical overview, geometric characterization and visualization of its fundamental properties. The material may serve as an introduction to the rich field of triangle geometry. The language of the article is intentionally kept accessible, aiming to engage and inspire interest in the topic among secondary school students.

Keywords: Feuerbach hyperbola, trilinear coordinates, triangle geometry, locus, geogebra

Kľúčové slová: Feuerbachova hyperbola, trilineárne súradnice, trojuholníková geometria, množina bodov, geogebra

1 Predslov

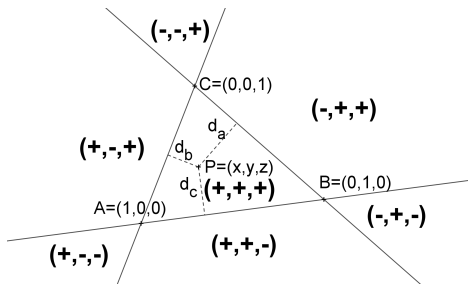
Trojuholníková geometria zažívala svoju prvú "zlatú éru" v druhej polovici 19. storočia, keď matematici ako Émile Lemoine, Henri Brocard či Joseph Neuberg objavovali a skúmali množstvo nových bodov, priamok a kužeľosečiek spojených s trojuholníkom. Záujem o trojuholníkovú geometriu však časom postupne klesol, čiastočne kvôli vnímaniu predmetu ako elementárneho a vizuálne náročného, ako aj pre konkurenciu s inými atraktívnymi témami geometrie. Po období útlmu v priebehu 20. storočia nastal koncom storočia nový rozmach záujmu, výrazne podporený rozvojom počítačovej geometrie. V jeho popredí stojí najmä Clark Kimberling, autor rozsiahlej Encyclopedia of Triangle Centers (ďalej označený skratkou ETC [1]), ktorá systematizuje a sprístupňuje výsledky súčasného výskumu a prispieva k renesancii tejto oblasti geometrie.

Medzi najznámejšie trojuholníku opísané hyperboly (teda hyperboly, ktoré prechádzajú vrcholmi trojuholníka) patria Kiepertova, Feuerbachova a Jeřábekova hyperbola. V našom príspevku sa zameriavame na bohatú históriu Feuerbachovej hyperboly a jej geometrické vlastnosti. Ukážky rôznych konštrukcií a vlastností sú dostupné v interaktívnej Geogebra knihe: <https://www.geogebra.org/m/ahbug4j5>

2 Stručný prehľad trilineárnych súradníc

Na analýzu stredov a kuželosečiek spojených s trojuholníkom je možné využiť viacero súradnicových systémov, medzi ktoré patria trilineárne, barycentrické a polárne. V rámci tejto kapitoly sa zameriavame na **trilineárne (normálové) súradnice** a poskytujeme ich stručný prehľad. V ďalšej časti príspevku pod súradnicami budeme rozumieť trilineárne súradnice. Elegancia tejto súradnicovej sústavy spočíva v proporcionality, vďaka ktorej sa zjednodušuje počítanie so súradnicami. Pri riešení problémov sa zídu známe vzťahy, ako sínusová veta, kosínusová veta, Pytagorova veta a trigonometrické vzorce. Pri opise sme sa inšpirovali článkom o Kiepertovej hyperbole, napísaný matematikmi Eddy a Fritsch [2] a knihou od Sommerville [3].

Nech máme daný všeobecný referenčný trojuholník ABC s dĺžkou strán $|BC| = a$, $|CA| = b$, $|AB| = c$ a veľkosťou uhlov $|\angle BAC| = A$, $|\angle ABC| = B$, $|\angle BCA| = C$. Na označenie polomeru vpísanej kružnice budeme v ďalšom používať znak r a polomer opísanej kružnice je označený ako R .



Obr. 1: Orientovanosť trilineárnych súradníc.

Ľubovoľný bod P má (trilineárne) súradnice (x, y, z) vzhľadom na referenčný trojuholník, ak platí

$$x : y : z = d_a : d_b : d_c,$$

kde d_a je *orientovaná vzdialenosť* bodu P od strany BC . Orientovanosť znamená, že vzdialenosť je kladná, ak bod P leží v tej istej polrovine od strany BC ako stred vpísanej kružnice a vzdialenosť je záporná, ak leží v opačnej polrovine. Analogickým spôsobom určíme vzdialenosti d_b od strany AC a d_c od strany AB (obr. 1.). Vrcholy trojuholníka majú súradnice $A = (1, 0, 0)$, $B = (0, 1, 0)$ a $C = (0, 0, 1)$. Skutočné vzdialenosti bodu A od strán je $d_a = c \cdot \sin B$, $d_b = 0$, $d_c = 0$, odkiaľ vzniká pomer $x : y : z = d_a : d_b : d_c = 1 : 0 : 0$. Súvis medzi trilineárnymi súradnicami

bodú P a skutočnými vzdialenosťami tohto bodu od strán referenčného trojuholníka je daný rovnicou

$$\frac{d_a}{x} = \frac{d_b}{y} = \frac{d_c}{z} = \frac{r(a+b+c)}{ax+by+cz}.$$

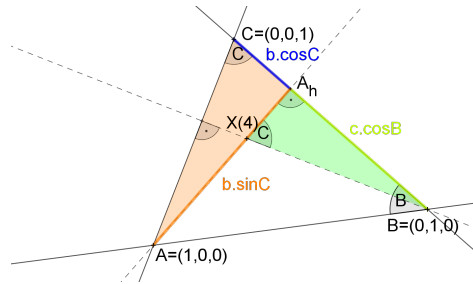
Stred vpísanej kružnice má triviálne súradnice $X(1) = (r, r, r) = (1, 1, 1)$.

Súradnice ortocentra trojuholníka sú $X(4) = (\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C})$. Vzdialenosť ortocentra od strany a je $d_a = |A_h X(4)|$ a dostaneme ju z podobných trojuholníkov $X(4)BA_h$ a CAA_h (Obr. 2.), teda z pomerov

$$\frac{d_a}{c \cdot \cos B} = \frac{b \cdot \cos C}{b \cdot \sin C},$$

analogicky pre vzdialenosti d_b a d_c . Pomer vzdialeností bodu $X(4)$ od strán referenčného trojuholníka je teda

$$\begin{aligned} x : y : z &= \frac{c}{\sin C} \cos B \cos C : \frac{a}{\sin A} \cos A \cos C : \frac{b}{\sin B} \cos A \cos B = \\ &= \frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}. \end{aligned}$$



Obr. 2: Výška trojuholníka.

Trilineárnu súradnicu ortocentra môžeme napísať aj v stručnom tvare $X(4) = \left(\frac{1}{\cos A}, \dots, \dots\right)$, keďže súradnice vyhovujú reprezentácii $f(A, B, C) : f(B, C, A) : f(C, A, B)$. Ortocentrum je teda stredom trojuholníka (podrobnejšie v [4]).

Súradnice ďalších stredov trojuholníka sa nachádzajú už v spomenutej ETC, je ich už vyše 70000.

Pri práci so súradnicami v tomto príspevku využijeme aj kollinearitu bodov. Tri rôzne body s trilineárnymi súradnicami (x_1, y_1, z_1) , (x_2, y_2, z_2) a (x_3, y_3, z_3) sú kollinéarne, ak platí

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

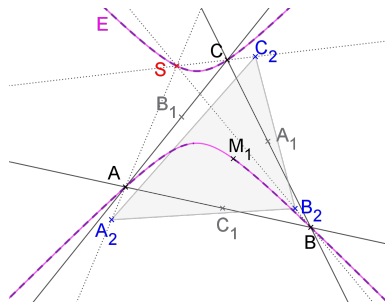
3 Feuerbachova hyperbola - historický prehľad

Po veľmi stručnom zavedení trilineárnych súradníc sa zameriavame na historický vývoj Feuerbachovej hyperboly a prostredníctvom geometrickej charakterizácie predstavíme spôsob práce s trilineárnymi súradnicami.

3.1 Lemoinov opis

Vôbec prvý článok o tejto hyperbole nájdeme v príspevku *Contributions a la géométrie du triangle* z roku 1889 napísaný matematikom **Émile Lemoine** [5]. Lemoine sformuloval vetu:

Nech sú dané štyri body M_1, A, B, C rovnoosej hyperboly E . Päť kolmíc z bodu M_1 na strany BC, CA, AB sú A_1, B_1, C_1 . Nech A_2, B_2, C_2 sú opačne orientované inverzie bodov A_1, B_1, C_1 , kde M_1 je stred a ρ je polomer. Potom trojuholníky ABC a $A_2B_2C_2$ sú v perspektívnej kolíneácii so stredom na hyperbole E pre všetky hodnoty ρ . (obr. 3.) (Poznámka: opačne orientovaná inverzia je v pôvodnom článku označená ako transformácia recipročných lúčov a perspektívna kolíneácia ako homológia.)



Obr. 3: Trojuholníky v perspektívnej kolíneácii.

Ak M_1 je ťažisko, E je zhodná s Kiepertovou hyperbolou. Keď M_1 je stredom vpísanej kružnice, rovnica hyperboly je

$$\frac{\cos B - \cos C}{x} + \frac{\cos C - \cos A}{y} + \frac{\cos A - \cos B}{z} = 0, \quad (1)$$

ktorá je dnes už známa ako Feuerbachova hyperbola. Stredom hyperboly je bod dotyku vpísanej kružnice s kružnicou deviatich bodov, a má súradnice

$\sin^2(\frac{B-C}{2}) : \dots : \dots$. (Pozn.: Kružnica deviatich bodov je Feuerbachova kružnica.) Ak M_1 je stred jednej z vonkajších dotykových kružníc, rovnica hyperboly je

$$\frac{\cos B - \cos C}{x} - \frac{\cos A + \cos C}{y} + \frac{\cos A + \cos B}{z} = 0 \quad (2)$$

a stredom hyperboly je bod dotyku tejto kružnice s kružnicou deviatich bodov. Pre ďalšie stredy vonkajších dotykových kružníc sú rovnice analogické. V prípade, že M_1 je ortocentrum, hyperbola sa redukuje na tento bod.

3.2 Boutinov opis

Podrobnejší opis sa nachádza v článku *Sur un groupe de quatre coniques remarquables du plan d'un triangle* z roku 1890 napísaný matematikom **Auguste Boutin** [6].

Nech sú O, I, I_1, I_2, I_3 stredom opísanej kružnice, stredom vpísanej kružnice a stredmi vonkajších dotykových kružníc referenčného trojuholníka. Izogonálnou transformáciou priamok OI, OI_1, OI_2, OI_3 dostaneme štyri pozoruhodné kužeľosečky, označíme ich $B_I, B_{I_1}, B_{I_2}, B_{I_3}$. Tieto kužeľosečky sú opísané referenčnému trojuholníku, prechádzajú ortocentrom H , sú to teda štyri rovnostranné hyperboly.

Rovnica hyperboly B_I je v predchádzajúcej časti napísaná rovnica (1), hyperboly B_{I_1} je (2), a analogicky pre B_{I_2} a B_{I_3} . Priamky OI, OI_1, OI_2 a OI_3 sú dotyčnice k príslušným kužeľosečkám. Boutin uvádza aj súradnice stredov hyperbol $C_I, C_{I_1}, C_{I_2}, C_{I_3}$. Tieto štyri stredy sa nachádzajú na obvodě Feuerbachovej kružnice trojuholníka ABC , aj na priamkach $O_9I, O_9I_1, O_9I_2, O_9I_3$, kde O_9 je stredom Feuerbachovej kružnice.

Boutin našiel aj ďalšiu konštrukciu kužeľosečky B_I :

Nech sú body A_1, B_1, C_1 bodmi dotyku vpísanej kružnice so stranami trojuholníka ABC . Ak na IA_1, IB_1, IC_1 prenosieme v rovnakom smere dĺžky $IA_2 = IB_2 = IC_2 = d$, tak priamky AA_2, BB_2 a CC_2 sa pretínajú v jednom bode Q_d . Množina bodov Q_d s meniacou sa hodnotou d je kužeľosečka B_I . Pre stredy vonkajších dotykových kružníc je konštrukcia analogická. Trilineárne súradnice bodov A_2, B_2 a C_2 sú

$$A_2 = (d - r, d \cdot \cos C + r, d \cdot \cos B + r),$$

$$B_2 = (d \cdot \cos C + r, d - r, d \cdot \cos A + r),$$

$$C_2 = (d \cdot \cos B + r, d \cdot \cos A + r, d - r).$$

Z predpokladu kollinearit bodov A, Q_d, A_2

$$\begin{vmatrix} 1 & 0 & 0 \\ x & y & z \\ d - r & d \cdot \cos C + r & d \cdot \cos B + r \end{vmatrix} = 0,$$

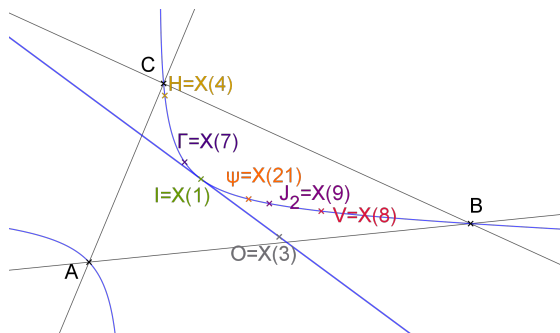
a podobne z kollinearity bodov B, Q_d, B_2 a bodov C, Q_d, C_2 dostaneme trilineárne súradnice bodu Q_d

$$Q_d = \left(\frac{1}{d \cdot \cos A + r}, \dots, \dots \right).$$

Elimináciou parametra d dostaneme rovnicu hyperboly (1).

Hyperbola B_I prechádza aj bodmi v pôvodnom článku označených ako v, Γ, I, J_2 . Boutin uvádza hodnoty parametra d , ktoré zodpovedajú týmto bodom, ich označenie predstavujeme aj v podobe, ako ich môžeme nájsť v ETC (obr. 4):

- pre A, B, C je $d = -\frac{r}{\cos A}, -\frac{r}{\cos B}, -\frac{r}{\cos C}$;
- pre $I = X(1)$, stred vpísanej kružnice, je $d = 0$;
- pre $\Gamma = X(7)$, Gergonne-ov bod, je $d = r$;
- pre $H = X(4)$, ortocentrum, je $d = \infty$;
- pre $v = X(8)$, Nagelov bod, je $d = -r$;
- pre $J_2 = X(9)$, Mittenpunkt, platí $\frac{1}{d} + \frac{1}{2R} + \frac{1}{r} = 0$;
- pre $\Psi = X(21)$, Schifflerov bod, platí $\frac{1}{d} + \frac{1}{r} + \frac{1}{R} = 0$.



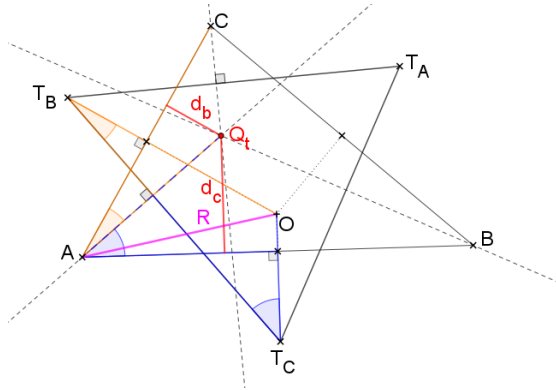
Obr. 4: Niektoré stredy nachádzajúce sa na Feuerbachovej hyperbole.

3.3 Mandartov opis

V časopise Mathesis z roku 1893 je ďalší hlavný článok o tejto hyperbole a bol napísaný z ručne písanej poznámky pána **H. Mandart** [7]. V poznámkach je uvedený, že Feuerbachovu hyperbolu si všimol pán Lemoine a pán Boutin. V článku hyperbola už nesie meno **Feuerbachova hyperbola** a je uvedená jej ďalšia konštrukcia:

Nech máme body T_A, T_B a T_C také, že T_A sa nachádza na osi strany BC , T_B na osi strany CA a T_C na osi strany AB , nech vzdialenosť týchto bodov od príslušných strán je t a nech sú orientované rovnakým smerom

zhľadom na trojuholník. Kolmice z bodov A , B , a C na zodpovedajúce strany $T_B T_C$, $T_C T_A$ a $T_A T_B$ majú práve jeden spoločný bod, označme Q_t . S meniacou sa hodnotou t je množina týchto bodov Feuerbachova hyperbola.



Obr. 5: Znázornenie súvislostí pri hľadaní súradníc bodu Q_t .

Pri hľadaní trilineárnych súradníc bodu Q_t (obr. 5.) využijeme, že vzdialenosť bodu Q_t od strany AC je $|AQ_t| \cdot \sin \angle Q_t AC$, od strany AB je $|AQ_t| \cdot \sin \angle Q_t AB$. Uhly $Q_t AC$ a $Q_t AB$ sú kolmé na zodpovedajúce uhly $OT_B T_C$ a $OT_C T_B$, z toho vyplýva pomer

$$\frac{d_b}{d_c} = \frac{y}{z} = \frac{\sin \angle Q_t AC}{\sin \angle Q_t AB} = \frac{\sin \angle OT_B T_C}{\sin \angle OT_C T_B} = \frac{OT_C}{OT_B} = \frac{R \cos C + t}{R \cos B + t}.$$

Parameter t má kladné znamienko, pretože body T_A, T_B a T_C sú namerané od stredu opísanej kružnice smerom von z trojuholníka. Z týchto pomerov získame trilineárne súradnice bodu Q_t

$$x : y : z = \frac{1}{R \cos A + t} : \frac{1}{R \cos B + t} : \frac{1}{R \cos C + t}. \quad (3)$$

Elimináciou parametra t z rovnice (3) dostaneme rovnicu hyperboly (1).

Mandart si všimol, že medzi parametrami t a d je nepriama úmera. Vzťah medzi parametrami získame z predpokladu $Q_d = Q_t$ a kollinearitu bodov A, A_2 a Q_t , teda

$$\begin{vmatrix} 1 & 0 & 0 \\ d-r & d\cos C+r & d\cos B+r \\ \frac{1}{R\cos A+t} & \frac{1}{R\cos B+t} & \frac{1}{R\cos C+t} \end{vmatrix} = 0,$$

z čoho dostaneme

$$d = \frac{rR}{t}.$$

Z konštrukcie trojuholníka $T_AT_BT_C$ je hneď zrejmé, že smery asymptôt hyperboly sú dané predpokladom, keď sú body T_A , T_B a T_C kollinearne. Kollinearita nastane pre hodnotu parametra

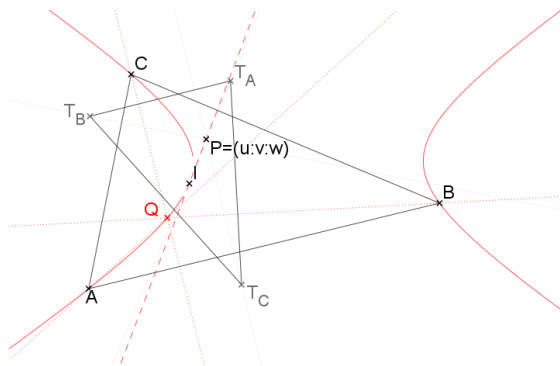
$$t = \frac{-R \pm \sqrt{R^2 - 2Rr}}{2}.$$

3.4 Gibertove zovšeobecnenie Feuerbachovej hyperboly

V roku 2004 pán Gibert [8] analyzoval *t-Mandartove trojuholníky* $T_AT_BT_C$ a Feuerbachovu hyperbolu charakterizoval na základe Mandartových výsledkov. Zovšeobecnil Mandartovu konštrukciu - pri konštrukcii bodov T_A , T_B a T_C nahradil stred opísanej kružnice všeobecným bodom $P = (u, v, w)$ (obr. 6.):

Nech je daný trojuholník ABC a ľubovoľný bod P . Na kolmice z bodu P na strany trojuholníka ABC zostrojme v rovnakom orientovaní body T_A , T_B a T_C tak, že vzdialenosti bodov T_A od strany BC , T_B od strany AC a T_C od strany AB sú t . Kolmice z vrcholov A , B a C na príslušné strany trojuholníka $T_AT_BT_C$ sa pretínajú v jednom bode $Q = \left(\frac{a}{at+2\Delta u}, \dots, \dots\right)$, kde Δ je obsah trojuholníka. S meniacou sa hodnotou t množina bodov Q je trojuholníku opísaná hyperbola, ktorá je izogonálnou transformáciou priamky IP . Rovnica hyperboly je

$$\frac{a(cv - bw)}{x} + \frac{b(aw - cu)}{y} + \frac{c(bu - av)}{z} = 0.$$

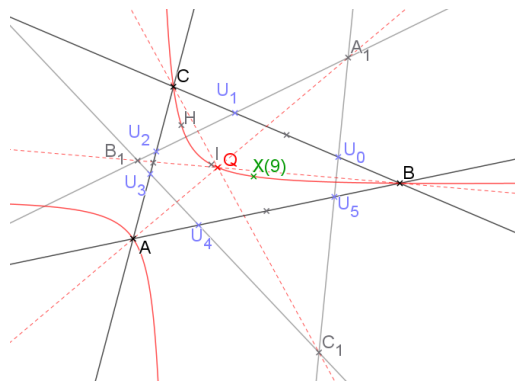


Obr. 6: Zovšeobecnená Mandartova hyperbola.

3.5 Kouřilová a Röscher

V roku 2013 pani Kouřilová a pán Röscher [9] napísali ďalšiu konštrukciu Feuerbachovej hyperboly:

Na jednej strane trojuholníka ABC zvolíme bod U_0 , nech je na strane BC a vo vzdialenosti m od vrcholu B . Stredovou súmernosťou bodu U_0 cez stred tejto strany dostaneme bod U_1 . Osovou súmernosťou bodu U_1 cez os uhla C dostaneme bod U_2 . Stredovou súmernosťou bodu U_2 cez stred strany AC dostaneme bod U_3 , atď. Cyklicky sa dostaneme k pôvodnému bodu U_0 . Priamky U_1U_2 , U_3U_4 a U_5U_0 definujú trojuholník $A_1B_1C_1$. Priamky AA_1 , BB_1 a CC_1 sa pretínajú v jednom bode, a s meniacou sa hodnotou m množina týchto bodov je Feuerbachova hyperbola (7. obr.).



Obr. 7: Konštrukcia cez stredové a osové súmernosti.

Autori skúmali aj analógiu tejto hyperboly na sférickom trojuholníku, jej opis však presahuje rozsah tohto príspevku.

4 Záver

Koncom 19. storočia bola trojuholníková geometria vnímaná ako elementárna a vizuálne náročná. Bariéra vizuálnej náročnosti dnes ľahko prekonáme s voľne dostupnými grafickými softvérmi a kvôli elementárnosti riešenie problémov môžu byť zvládnuteľné aj stredoškólakmi. Feuerbachova hyperbola sa ukázala byť všestranným objektom trojuholníkovej geometrie s viacerými rôznymi konštrukciami, preto je vhodným príkladom na demonštráciu práce s trilineárnymi súradnicami.

Podakovanie

Tento článok vznikol za podpory projektu KEGA 004UJS-4/2025 "Online interaktívne vzdelávacie prostredie na podporu prípravy študentov učiteľstva matematiky".

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Modeling 3D environments for virtual reality

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Abstract. The work focuses on creating a virtual environment, in which we demonstrate the use of various types of geometric surfaces in architecture. A key output is an interactive application. It allows users to view the 3D form of architectural concepts that are often perceived only in 2D. This 3D visualization contributes to more efficient learning and a better understanding of the spatial properties of the studied surfaces. The work also includes a description of the development processes and optimization of VR applications in the Unity environment. Although we primarily focus on the Meta Quest 2, the described principles are also applicable to other modern VR headsets. Given the rapid development and wide use of virtual reality, it is important to become familiar with the specifics of development for this platform.

Keywords: Unity, virtual reality, Meta Quest

1 Virtual reality

Virtual reality (VR) is a computer-generated, simulated experience that creates an immersive, three-dimensional (3D) environment. Using specialized hardware, like a head-mounted display (HMD) and motion controllers, it replaces your real-world view with a digital one. The goal is to trick your brain into believing you're physically present in the virtual world.

VR systems work by stimulating a user's senses to create the illusion of being in a different environment. This is achieved through a combination of hardware and software.

1.1 Components of VR headset

Today's widely available VR headsets (or VR goggles) consist of a head-mounted display (HMD) and a pair of controllers. Inside the HMD, there are one or more often two screens that project stereoscopic images. Another essential component of the headset is a series of lenses, as the display is located too close

for a person to focus on it. The headset contains various sensors to determine the user's head position and tilt. This typically includes gyroscopes (to detect orientation and tilt), accelerometers (to detect displacement), and cameras. The headsets include built-in speakers near the ears or headphones with stereophonic playback, also known as spatial audio, to simulate various real-world sounds. Adjustable straps, soft padding, and good weight distribution of the device. Controllers or gloves for tracking hand movements and capturing user input.



Fig. 1: VR headset

From the perspective of their connection to another device, we can divide them into two main types:

- Headsets that require a connection to a computer or a game console.
- The second category consists of standalone headsets.

1.2 Use of Virtual Reality and XR

The most common applications of virtual reality (VR) and XR are in video games, in medicine - offering surgery simulations for surgeons, in real estate sales, architecture, and interior design - allowing for a realistic preview of a room or building, in the training of police officers and soldiers - using simulations of crisis situations, allowing them to try them out "in real life" without risking health, life, or property [1], in aviation - advanced augmented reality systems are used when piloting fighter jets to simplify machine control. During the Covid-19 pandemic, "virtual tourism" expanded, meaning visiting various places and monuments without leaving one's residence. These technologies already have their place in many industries today, and this list continues to expand. [1]

1.3 The history of VR headsets

The absolute beginning can be considered the invention of the stereoscope by Charles Wheatstone. This device placed a slightly different two-dimensional image in front of each eye. The brain processed them into a single three-dimensional sensation, creating the illusion of depth. While not virtual reality in the modern sense, it was the first step toward tricking the senses to create a 3D

experience. The American cinematographer Morton Heilig designed and later built in 1957 a device called the Sensorama. This machine was designed to stimulate multiple senses at once—sight, sound, smell, and even touch through a vibrating chair. Although it wasn't a headset, it was an important concept focused on fully immersing the user in an experience. The Sword of Damocles by Ivan Sutherland in 1968 is considered the first true VR headset. The device was so heavy that it had to be suspended from the ceiling and hung in front of the user (hence the name "Sword of Damocles"). Although it only displayed simple wireframe models, it tracked head movement and changed the view, creating the first interactive 3D experience. In the 80s, NASA was developing the Virtual Interactive Environment Workstation (VIEW), which was intended for training astronauts. It included a headset and a pair of gloves used for interacting with the environment, and it applied the LEEP system with some modifications.

2 Unity

There are multiple environments available for developers of VR applications. One of the best-known and frequently used is Unity, which is used for developing games for various platforms. It offers tools for creating 2D/3D environments as well as XR environments, along with many existing models freely available for download in the Unity Asset Store.

2.1 Unity User Interface

When we open a new project in Unity, we are presented with the user interface, as shown in Figure 2.

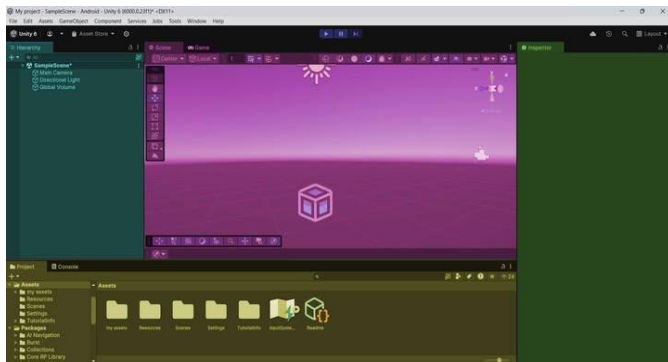


Fig. 2: Unity

The interface consists of 4 main windows: Scene - this is the main workspace where the game content is displayed and edited. Hierarchy - here,

we can see the objects that are currently in the scene. This refers to the Hierarchy window in Unity. Inspector - when you click on an object, its details are displayed here. You can then edit its properties or add the components mentioned previously. Project – it is used to browse all the files that are currently in the project. Here we will get to the models, materials, scripts and everything else that we have downloaded, created or was provided by Unity to build the game.

Then, proceed as follows:

1. Open a new project using New project and select the 3D core template.
2. In this project, in the Edit tab, select Project Settings and then choose XR Plugin Management, where you will click Install Plugin Management.
3. In the XR Plugin Management window, select the computer icon and check OpenXR. Do the same for the Android icon. In the newly opened OpenXR window, you need to add Oculus Interaction Profile under Enabled Interaction Profiles.
4. Finally, when you want to build the application, in File > Build Profiles under the Android section, click Switch platform and then select Quest 2 under Build Device. This step can be skipped if you want to run the result from your computer.

Clicking Build and run will compile the project (compilation, scene assembly, generation of the executable package, e.g., *.apk* or *.exe* etc.), save it to the selected location, and run.

3 Blender

Blender is open-source software used for modeling and animating 3D objects and environments. It is available for free, and its mission was to make 3D modeling accessible to everyone. Using the Python programming language, it is possible to create custom scripts within it to automate tasks or create objects.

3.1 Import models from Blender

1. We can download the models from Blender in formats such as .fbx or .obj.
2. This file can then be inserted into the Project window in Unity, and from there we can place it into the scene.
3. When we place the building into the environment: We delete unused elements (especially lights and cameras) and adjust its position by scaling/rotating/moving. Then we assign a Box Collider or Capsule Collider component in the Inspector and check the Provide Contact box, so that the player cannot pass through the walls. In the case of more complex geometry, we directly select the object that should have contact and assign it a Mesh Collider. For the object, it is worthwhile to select Extract Materials in the Inspector section, which will allow us to manipulate its materials directly in Unity.

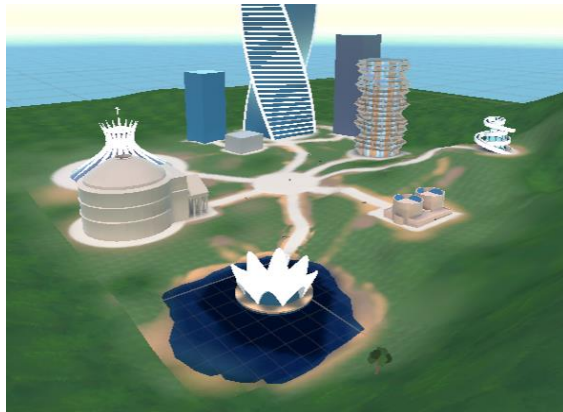


Fig. 3: Unity with objects from Blender

4 Conclusion

In this work, we focused on creating a standalone application that can be run on VR headsets. The resulting application offers a virtual tour of several buildings from around the world, showcasing the use of geometric surfaces in architecture. This work contributes to a better understanding of geometric surfaces in architecture through an interactive VR visualization. It allows for a more intuitive

perception of the spatial properties of these surfaces, which is often limited in traditional 2D sketches.

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Složenka

Procvičování prostorové představivosti hrou

Searching for matching pieces in a 3D puzzle

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Abstract. We present a web-based application designed to develop students' spatial reasoning skills. In this task, students are required to determine which of the presented 3D components can be assembled into a given target shape. Both the individual components and the target shape are visualized in a 3D interactive environment. Students can rotate the pieces, but only one piece is visible at a time, and the preview offers no orientation clues. This design encourages learners to actively engage their spatial imagination, requiring them to mentally manipulate, retain, and integrate visual information to assess whether and how the components can form the complete structure.

Keywords: spatial skills, 3D interactive web, mental 3D manipulation

Klíčová slova: Prostorová představivost, interaktivní trojrozměrná scéna, mentální manipulace v prostoru

1 Prostorová představivost

V literatuře najdeme různé definice pojmu „prostorová představivost“, např. [1]. Obvykle se jedná o výčet určitých schopností jako jsou vnímání prostorových objektů, mentální manipulace a orientace v prostoru. Prostorová představivost je nezbytná pro každého člověka (i zvíře), aby se dokázal orientovat ve světě kolem sebe, ale jako učitele geometrie nás zajímá jako schopnost využívaná k řešení problémů a úloh.

Vytváření softwarových nástrojů na podporu a rozvíjení prostorové představivosti se věnujeme už dlouho, viz [2], [3], [4], a v tomto textu chceme představit další z nich, který jsme pojmenovali „Složenka“.

2 Návrh aplikace na procvičování prostorové představivosti

Naším cílem bylo vytvořit aplikaci, která by uživatele (žáky) vedla k tomu, aby si představovali a pamatovali trojrozměrné objekty a jejich transformace.

Taková aplikace by představovala určité dané prostorové objekty a kladla by uživateli otázku, zda složením těchto objektů (dvou nebo více) může vzniknout požadovaný výsledný objekt. Například mějme objekty složené ze tří slepených krychlíček (Obr. 1):



Obr. 1: Výchozí objekt „L“

Takové objekty můžeme libovolně otáčet a posouvat a ptáme se, zda z nich lze složit krychli rozměru $2 \times 2 \times 2$.

Odpověď na tuto otázku nevyžaduje mnoho prostorové představivosti, stačí si spočítat, že krychle $2 \times 2 \times 2$ obsahuje 8 krychliček a z dílků po třech krychličkách ji složit nelze. Ale když se zeptáme, zda z daných dílků lze složit krychli rozměru $3 \times 3 \times 3$, odpověď už tak zřejmá není.

2.1 Požadavky

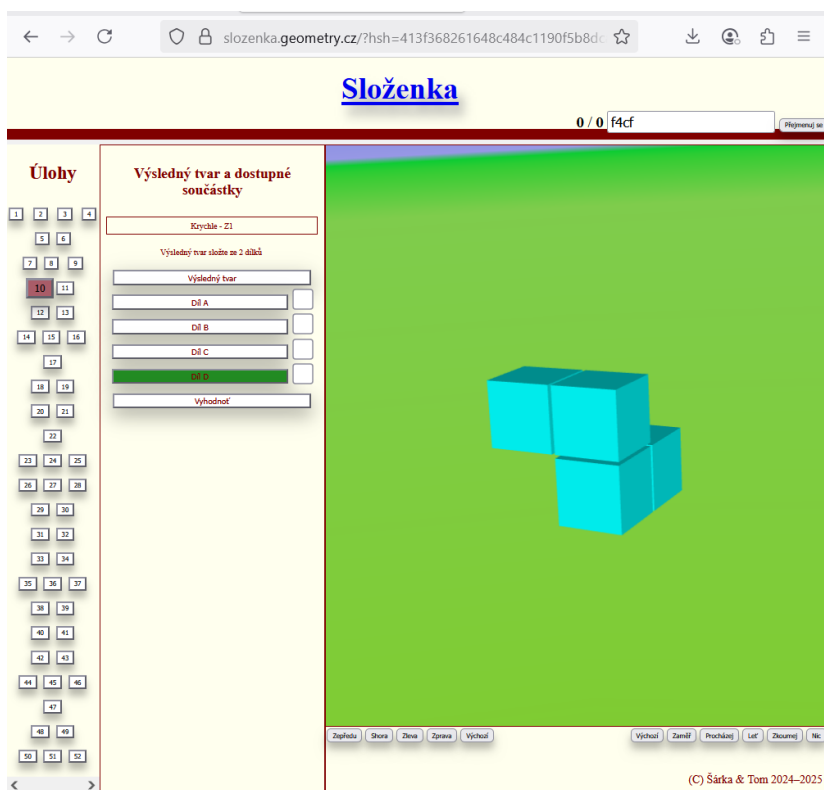
Požadavky na navrhovanou aplikaci byly tyto:

- budou představeny jednotlivé dílky a úkolem bude označit všechny dílky, ze kterých lze složit požadovaný výsledný tvar
- dílky i výsledný tvar si bude moci uživatel libovolně prohlížet, zvětšovat, posouvat a otáčet je, aby si je mohl dobře prohlédnout, vrátet se k nim
- ...vždy ale uvidí pouze jeden dílek nebo výsledný tvar, takže bude nucen si tvary pamatovat, případně v duchu otáčet
- aplikace bude obsahovat větší počet úloh, mezi kterými si uživatel bude moci vybírat
- aplikace bude hodnotit, jestli je odevzdaná odpověď správná
- budou označeny úlohy, které už uživatel vyřešil, i ty, které zkusil řešit a neuspěl
- informace o řešených úlohách zůstanou zachovány i při opakovaném spuštění programu
- aplikace nebude vyžadovat žádný login, registraci, heslo apod.
- aplikace nebude řešit vztah učitel–žák ve smyslu, že by učitel zadával úlohy žákům a měl možnost vidět jejich výsledky; bude si ji moci pustit kdokoli a bude pro všechny obsahovat stejné úlohy
- aplikace bude mít tvar webové stránky, takže nebude potřeba nic stahovat ani instalovat a spouštět.

2.2 Výsledná aplikace

Výsledný program má podobu webové stránky dostupné na adrese <https://slozenka.geometry.cz/>. Při vstupu na tuto stránku se automaticky vygeneruje unikátní přihlašovací kód a uživatel je přesměrován na adresu s tímto kódem. Když si tuto adresu uloží do záložek, dostane se vždy ke svému seznamu vyřešených úloh.

Vlastní stránka aplikace obsahuje nadpis, který je zároveň odkazem na nápovědu, seznam úloh s vyznačením, které úlohy už byly řešeny a vyřešeny, a vedle něj zadání vybrané úlohy s tlačítky pro zobrazení jednotlivých tvarů tvořících zadání této úlohy. Po vybrání dílku se tento dílek zobrazí v poli zobrazení, kde s ním může uživatel pohybovat a otáčet (Obr. 2). V poli se zadáním úlohy může zaškrtnout dílky, ze kterých lze podle jeho názoru složit výsledný tvar. Po stisku tlačítka „Vyhodnot“ se dozví, zda byla odpověď správná – a pokud ano, uvidí ve 3D scéně i požadovaný tvar složený z odpovídajících dílků.

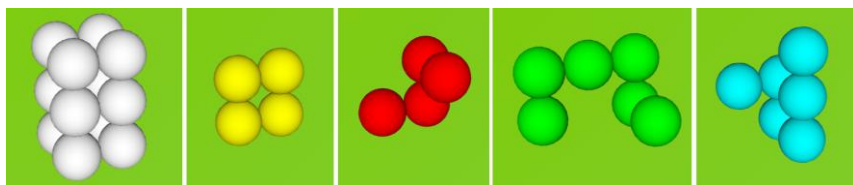


Obr. 2: Prostředí aplikace

2.3 Příklady úloh

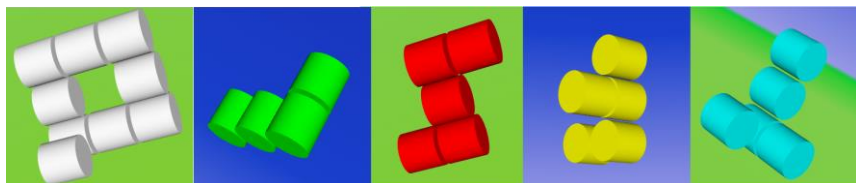
Výsledný tvar i dílčí tvary jsou vždy viditelně složeny z „jednotkových“ (tj. základních shodných) krychliček, případně válců či koulí téže velikosti.

Kromě úvodních, záměrně velmi snadných úloh, které jsou určeny k seznámení s aplikací, jsou v seznamu zařazeny další snadné úlohy, k jejichž vyřešení stačí spočítat základní krychličky v dílku či posoudit výsledný rozměr celku. Takové úlohy nevyžadují pokročilou prostorovou představivost. Příklad takové úlohy vidíme na Obr. 3. Dílky jsou zde složeny z kuliček, což je pro vnímání celku obtížnější než dílky složené z krychliček.



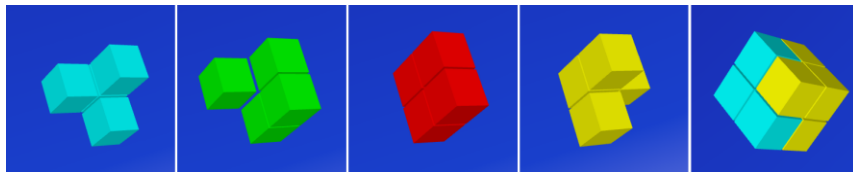
Obr. 3: Úloha, v níž k vyřešení stačí spočítat kuličky, celek jich má 12

V další úloze už počítání jednotkových dílků (zde jsou to válce) nestačí, výsledný tvar je složen z devíti základních válců, takže k jediné čtveřici je třeba hledat odpovídající pěti.



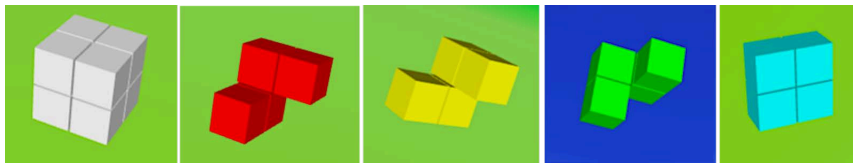
Obr. 4: Úloha z dílčích válců

V úloze na Obr. 5 má každý dílek 4 jednotkové krychličky. Vpravo je vidět výsledný složený tvar, který žák uvidí po vyhodnocení správného řešení.



Obr. 5: Jednotlivé dílky a výsledné složení požadovaného tvaru

V některých úlohách je třeba rozlišovat mezi nepřímo shodnými dílky. Příklad je na Obr. 6.



Obr. 6: Obtížná úloha – zelený dílek je nepřímo shodný s červeným i žlutým.

3 Zkušenosti

Složenku jsme nechali řešit různě staré žáky a studenty gymnázia: 13leté žáky sekundy, 15leté žáky kvarty osmiletého gymnázia a studenty maturitního ročníku kurzu deskriptivní geometrie. Zajímalo nás především to, zda pro ně budou úlohy obtížné. Rozdíly v úspěšnosti a rychlosti řešení souvisely spíše s individuálními schopnostmi žáků, než s věkem. Zdatní řešitelé vyřešili celou sadu úloh za 30–40 minut. Nejrychleji, za 25 minut, celou sadu vyřešil žák sekundy a v sekundě také všichni až na jednoho vyřešili celou sadu během jedné vyučovací hodiny. Nejvíce žáků, kteří sadu za 45 minut nevyřešili, bylo v kvintě.

3.1 Strategie řešení

Vzhledem k tomu, že dílky jsou ve výchozí poloze různě otočené (tedy ne tak, jak patří do výsledného tvaru) a ve scénách nejsou žádné prvky pro orientaci v prostoru, museli si žáci sami najít způsob, jak si tvar částí zapamatovat a jak rozhodnout o výsledku. Starší žáci a studenti počítali základní kostičky v dílech A, B, C, D, což v některých úlohách pro rozhodnutí o výsledku stačilo. Jindy odhadli dílek, který by mohl do výsledku patřit (největší dílek, případně jen první – dílek A) a určili k němu doplněk. Pokud doplněk nenašli, postup opakovali s dalším kandidátem. Dílky k výběru jsou vždy jen čtyři, postup tedy vedl rychle k cíli.

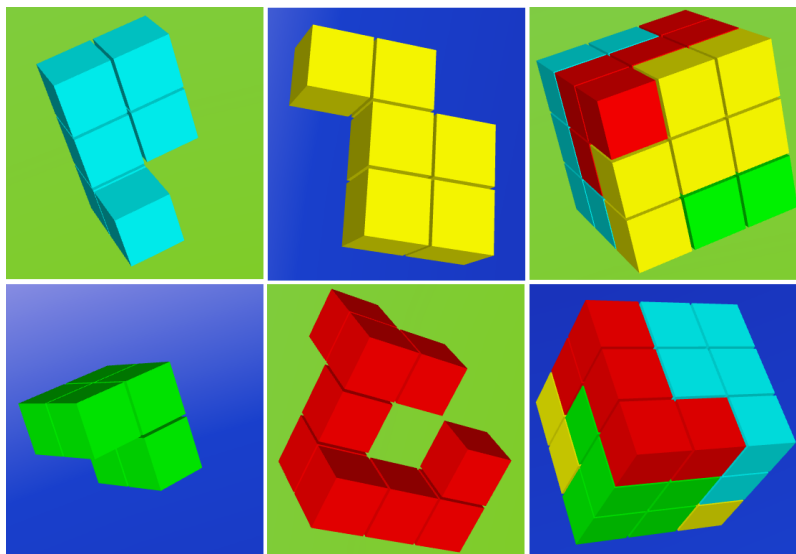
Žáci sekundy nedokázali slovně popsat strategii řešení, často uváděli, že si „pamatovali tvar“ (využívali tedy „fotografickou paměť“).

Podle očekávání byly nejobtížnější ty úlohy, kde bylo třeba vybrat správný z nepřímo shodných dílků. Tam žáci přiznávali náhodné pokusy.

Celkově žáci úkol hodnotili jako nepříliš obtížný, po skončení se většinou necítili unaveni.

4 Odpověď na závěr

Odpověď na úvodní otázku jsme uložili do posledních úloh sady. Z dílků „L“ krychli $3 \times 3 \times 3$ složit jde. Například složením všech dílků (v nichž jsou tvary „L“ je dobře vidět) na Obr. 7.



Obr. 7: Krychle $3 \times 3 \times 3$ z dílků „L“

Literatura

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Staň se na den vědkyní a vymodeluj si obalovou plochu

Become a woman-scientist for a day and explore envelope surface modelling

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Abstract. The paper provides a reflection on the event Become a Woman-Scientist for a Day 2025, held on February 11 at the Czech Technical University in Prague on the occasion of the International Day of Women and Girls in Science. The event, aimed at highlighting the essential role of women in the scientific community and promoting their access to scientific education, attracted considerable interest, particularly among secondary school students. The authors contributed to the event with a lecture on engineering applications of envelope surfaces, followed by a workshop. During the modelling of envelope surfaces in CAD, the significant didactic potential of a self-created dynamic 3D model became evident, as it effectively illustrated a rather complex theory unfamiliar to the participants.

Keywords: Envelope surface, characteristic of envelope surface, CAD model, Rhinoceros & Grasshopper

Klíčová slova: Obalová plocha, charakteristika obalové plochy, CAD model, Rhinoceros & Grasshopper

1 Úvod

Akce Staň se na den vědkyní pořádaná na Českém vysokém učení technickém v Praze (ČVUT) při příležitosti Mezinárodního dne žen a dívek ve vědě (11. únor) zajišťuje již několikátým rokem program pro dívky zájímající se o budoucí kariéru ve vědě a ukázky vědeckých prací, jakými se ženy ve vědě na ČVUT zabývají. Během dne si účastnice mohly poslehnout přednášky odbornic z akademické sféry, na které navazovaly praktické ukázky a možnost vyzkoušet si vědu na vlastní kůži v rámci témat, která sahala od částicové fyziky a robotiky až po aplikace geometrického modelování ve strojírenství. Na organizaci se podílela Fakulta strojní spolu s Fakultou jadernou a fyzikálně inženýrskou a Fakultou elektrotechnickou ČVUT.

V rámci příspěvku Fakulty strojní k této akci zazněla v dopoledním bloku přednáška na téma Geometrie obalových ploch a jejich aplikace ve

strojírenství, na kterou odpoledne navazovalo cvičení zaměřené na geometrickou aplikaci obalových ploch v numericky řízeném obrábění. Studentky byly seznámeny s výzvou, kterou tato úloha představuje: určit obalovou plochu vytvořenou pohybem obecné rotační plochy není triviální a současné CAD (Computer Aided Design) systémy nedisponují příkazy pro řešení tohoto zadání. Proto byla na Ústavu technické matematiky Fakulty strojní vyvinuta unikátní metoda modelování obalových ploch založená na aplikaci poznatků z diferenciální geometrie. Účastnicím byla tato metoda vysvětlena a měly poté možnost vytvořit si dle vlastního návrhu svůj originální CAD model obalové plochy v programu Rhinoceros s využitím vestavěného grafického programátorského prostředí Grasshopper.

2 Obalová plocha

Definice 1: Obalová plocha je část povrchu $\mathbf{E}(s, t)$ tělesa $\mathbf{B}(u, v, t)$, které vznikne jednoparametrickým pohybem tvořící plochy $\mathbf{S}(u, v)$ po trajektorii $\mathbf{T}(t)$, jestliže platí (u, v, s, t) jsou reálné parametry z uzavřených intervalů)

- Obalová plocha $\mathbf{E}(s, t)$ a každá uv -parametrická plocha tělesa $\mathbf{B}(u, v, t)$ se dotýkají podél parametrické s -křivky plochy $\mathbf{E}(s, t)$, která se nazývá charakteristika obalové plochy.
- V každém bodě obalové plochy $\mathbf{E}(s, t)$ existuje společná tečná rovina této plochy a jediné parametrické uv -plochy tělesa $\mathbf{B}(u, v, t)$.
- Neexistuje plocha, která by byla současně částí obalové plochy $\mathbf{E}(s, t)$ a některé parametrické uv -plochy tělesa $\mathbf{B}(u, v, t)$. [1]

Většina vysokoškolských sylabů obsahuje syntetické řešení nalezení charakteristiky obalové plochy generované přímočarým, rotačním a šroubovým pohybem roviny, koule a specificky zadané rotační plochy (meridián je tvořen přímkami či kružnicovými oblouky), viz např. [2]. Pokud je uvedeno analytické řešení, je buď redukováno na implicitní vyjádření zadaných útvarů nebo je zvoleno takové zadání, že obalová plocha je předem známa. Příkazy v CAD systémech, které by umožňovaly modelovat obalovou plochu jinak než jako předem známý útvar, neexistují a v CAM (Computer Aided Manufacturing) systémech se obalové plochy generované pohybem nástroje aproximují plochami ekvidistantními ke jmenovitému povrchu.

Analytický přístup založený na podmínce, že charakteristika je množina bodů, ve kterých jsou komplanární tečné vektory k parametrickým křivkám tvořící plochy a tečný vektor k trajektorii uvažovaného bodu je uveden v [3]. Zcela zásadní význam v teorii obalových ploch zaujímá [4], kde je použit DG/K (Differential Geometry/Kinematics) přístup k získání analytické reprezentace obalové plochy v dostatečně obecném pojetí, a sice

na základě podmínky, že obalová plocha je plocha, která se dotýká jedno-parametrické soustavy tvořících ploch.

3 CAD model rotační obalové plochy

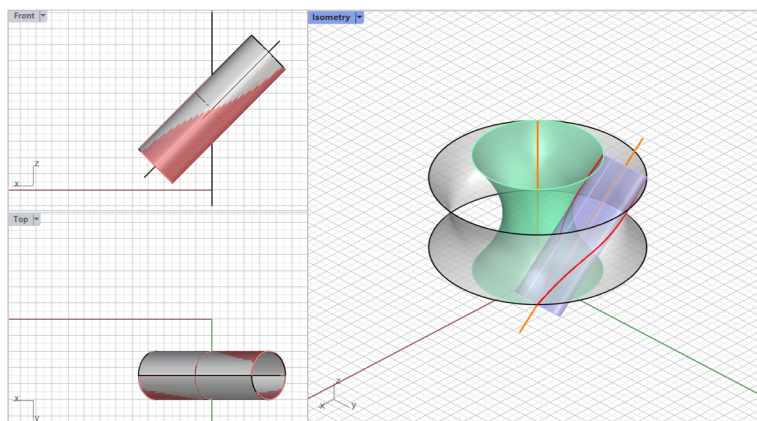
Definice 1 uvedená v [1] je zobecněním přístupu v [3] a [4] v tom smyslu, že obalová plocha je plocha v tělese generovaném jednoparametrickým pohybem (obecným) tvořící plochy (určené obecným meridiánem), v jejímž každém bodě jsou komplanární všechny tři tečné vektory k parametrickým křivkám tohoto tělesa. Z definice vyplývá také důležitá skutečnost, která je rozhodující pro vytvoření CAD reprezentace obalové plochy: uvažujeme-li dvě nekonečně blízké parametrické uv -plochy tělesa $\mathbf{B}(u, v, t)$, splynou charakteristiky v těchto polohách v jedinou křivku, a tudíž charakteristika je průniková křivka dvou nekonečně blízkých poloh tvořící plochy, jak je uvedeno v [2] (str. 37, vlastnost 4). Při přímočarém, rotačním a šroubovém pohybu má charakteristika neproměnný tvar, tudíž stačí vymodelovat jedinou průnikovou křivku dvou – kvůli konečné přesnosti CAD/CAM systémů – dostatečně blízkých poloh tvořící plochy a obalovou plochu poté vytvořit odpovídajícím pohybem takto získané charakteristiky [1]. Při obecném pohybu má charakteristika proměnný tvar, tudíž je nutné vymodelovat dostatečné množství dvojic blízkých poloh tvořící plochy a obalovou plochu vytvořit jako plochu procházející průnikovými křivkami.

3.1 Modelování obalových ploch na cvičení

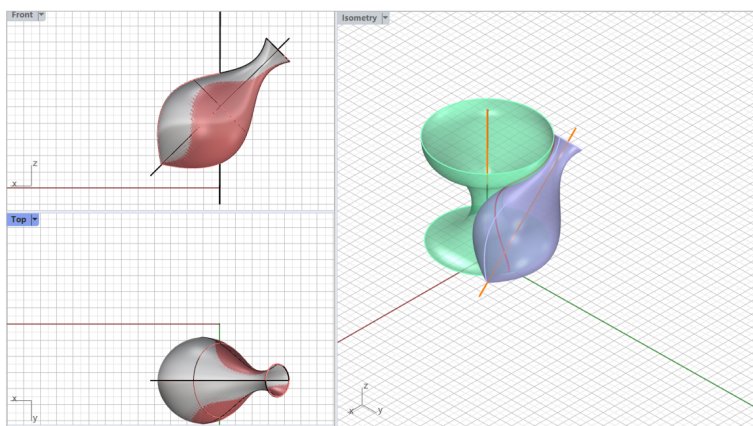
Princip tvorby CAD modelu rotační obalové plochy v Rhinoceros & Grasshopper je ukázán na obr. 1. Obalová plocha je zde generovaná rotací válcové rotační plochy kolem osy, která je s osou válcové plochy mimoběžná. V nárysu a půdorysu jsou barevně odlišeny dvě blízké polohy tvořící plochy. Jejich průniková křivka ještě není zkonstruována, nicméně již z barevného přechodu obou ploch je její tvar zřetelně patrný. V isometrickém pohledu je zobrazen hotový CAD model v poloprůhledném režimu.

V ukázkovém cvičení si nejprve účastnice v předem připravených šablonách pro Rhinoceros vytvořily statický CAD model obalové plochy z obr. 1, z něhož si poté v Grasshopperu vytvořily dynamický model, viz obr. 3. V dynamickém modelu bylo možné tvořící plochou pohybovat, a tím zřetelně demonstrovat princip vzniku obalové plochy a její vlastnosti. Studentky se také naučily měnit barvy a vlastnosti objektů a měnit typ zobrazení.

V další části cvičení následovala konstrukce obecné rotační tvořící plochy, kdy meridiánem byla B-spline křivka 4. stupně určená danými řídicími body, viz obr. 2.

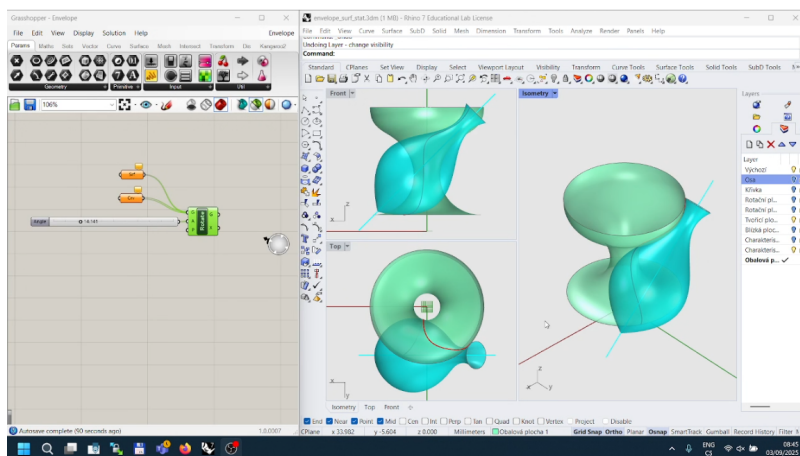


Obr. 1: Obalová plocha generovaná rotací rotační válcové plochy



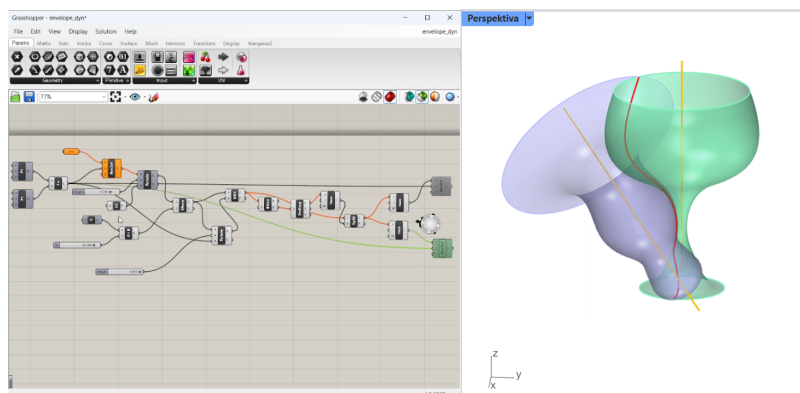
Obr. 2: Obalová plocha generovaná rotací obecné rotační plochy

V poslední části cvičení si účastnice vytvořily parametrický dynamický model rotační obalové plochy v Grasshopperu, viz obr. 4, ve kterém bylo možné měnit jak polohu řídících bodů meridiánu tvořící plochy, tak i její polohu vůči ose rotace. Studentky tuto závěrečnou část uvítaly s velkým nadšením, neboť dynamický parametrický CAD model umožňoval uplatnit jejich kreativní potenciál. Mnohé z nich ocenily toto interaktivní a praktické cvičení, které jim umožnilo seznámit se s problematikou obalových ploch. Hlavní důraz při svém hodnocení kladly na práci v Grasshopperu,



Obr. 3: Dynamický model rotační obalové plochy

kteřá jim poskytla nové dovednosti v oblasti parametrického designu a vizualizace, které jsou v současném vědeckém a inženýrském prostředí stále žádanější.



Obr. 4: Parametrický dynamický model obecné rotační plochy

4 Závěr

Akce Staň se na den vědkyní se těší čím dál většímu zájmu – letošního 9. ročníku se celkově zúčastnilo přes 140 studentek. V rámci této akce se během cvičení věnovanému modelování obalových ploch v CADu zřetelně projevil značný didaktický potenciál vlastnoručně vytvořeného parametrického dynamického 3D modelu k objasnění poměrně složité teorie obalových ploch, se kterou se studentky středních škol dosud nesetkaly. Získané dovednosti a první zkušenost s parametrickým modelováním tedy mohou studentky inspirovat k dalšímu vzdělávání v oblasti strojního inženýrství, což je jedním z hlavních cílů zmiňované akce.

Poděkování

Článek vznikl za podpory projektu SGS24/119/OHK2/3T/12 Pokročilé metody matematicko-geometrického modelování ve strojírenství.

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Group of isometries of hyperbolic plane

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Abstract. We sketch some basic properties and applications of isometries within hyperbolic plane. The properties are well known and the paper stresses the main lines of development of the area. We also point to some applications not only within the area of hyperbolic geometry. We would like to point out that many approaches from various fields of math meet in hyperbolic geometry and therefore it is a suitable topic where they can be learned following the basics.

Keywords: hyperbolic geometry, isometry, stereographic projection

1 Introduction

Non-Euclidean geometry was discovered by N. Lobachevsky and in parallel and in a different way by J. Bolyai around the end of the 3rd decade of the XIXth century. The key goal was to show that the parallel postulate of Euclid fails to be dependent on the other postulates. Hence, infinitely many parallel lines through a point (hyperbolic geometry) or no parallel line through a point (spherical geometry) might be a case. The goal was reached and a vast area of research has been revealed. In the hyperbolic case, the surface representing the plane has constant negative Gaussian curvature $K = -1$. The length of a circle grows exponentially with radius unlike the Euclidean or spherical cases. On the other hand, the isometries have similar structure in comparison with the Euclidean ones when constructing, though the group of isometries is much more complicated.

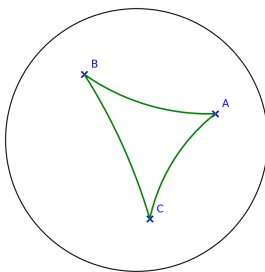


Fig. 1: A triangle in hyperbolic plane in Poincaré disc model.

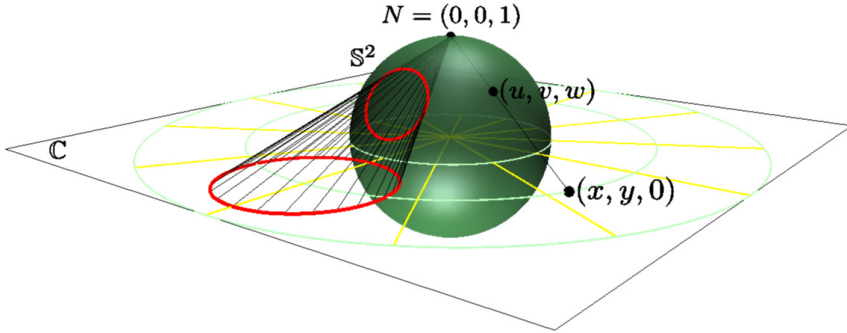


Fig. 2: Stereographic projection of the 2-dimensional sphere and a real plane (or complex line).

2 Stereographic projection

Stereographic projection connects points of a sphere and $\mathbb{C}$. This very useful mapping can be met in many areas as a basic fact/construction. The connection between coordinates (u, v, w) of a point of a sphere $u^2 + v^2 + w^2 = 1$ is mapped to the plane point $(x, y, 0)$ so that $x(u, v, w) = \frac{u}{1-w}$, $y(u, v, w) = \frac{v}{1-w}$ and the inverse mapping is given by $u = \frac{2x}{1+x^2+y^2}$, $v = \frac{2y}{1+x^2+y^2}$, $w = \frac{-1+x^2+y^2}{1+x^2+y^2}$. The mapping preserves angles and it maps circles on the sphere to circles or lines in the plane (see fig 2).

3 Models of hyperbolic geometry

There are many models of hyperbolic geometry, some of them just local. The advantage of so many models is that we can easily identify properties in one model and via isomorphisms use this information within the other model as well.

The primary model is a hyperboloid model $\mathbb{U}$ for $\mathbf{q}(x, y, z) = x^2 + y^2 - z^2$ for $z \geq 0$. Using appropriate projections, one obtains other models such as Poincaré disc model $\mathbb{U}_{\text{Poincare}}$ (improvement of Klein disc model $\mathbb{U}_{\text{Klein}}$), half-sphere model $\mathbb{S}_+^2$, halfplane model $\mathbb{U}_{\text{halfplane}}$ (see fig. 3, left) and there is also local pseudosphere model (see fig. 3, right).

All the models except the pseudosphere are interconnected via certain projections (more-less stereographic).

4 Metrics in hyperbolic geometry

The metrics can be computed by methods of differential geometry from the hyperboloid model, but for the other models, the formula of the hyperbolic distance can be easily found in the Klein's model for points $\mathbf{a}, \mathbf{b}$ with ideal

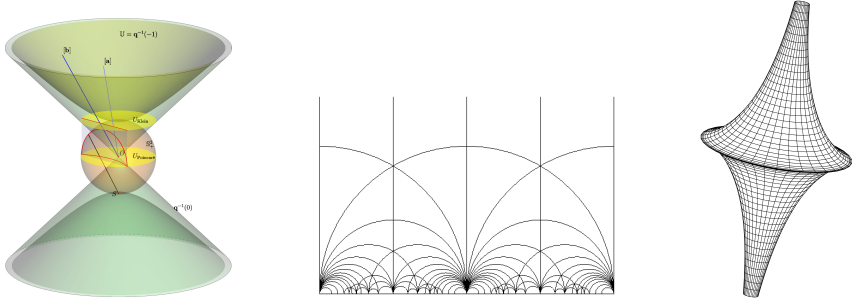


Fig. 3: Models of hyperbolic plane

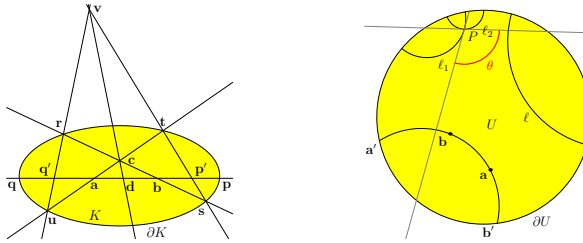


Fig. 4: (left) Cayley-Klein metric of a convex figure; (right) Hyperbolic lines and angles in Poincaré model.

points $\mathbf{a}'$, $\mathbf{b}'$ on the hyperbolic line $\mathbf{ab}$ based on Cayley-Klein metric (see fig. 4, left) applicable in any convex figure using Euclidean distances

$$d_H(\mathbf{a}, \mathbf{b}) = \ln \frac{d_E(\mathbf{b}', \mathbf{a})d_E(\mathbf{a}', \mathbf{b})}{d_E(\mathbf{b}', \mathbf{b})d_E(\mathbf{a}', \mathbf{a})}.$$

Connecting projective geometry, Euclidean geometry as well as hyperbolic geometry is done not only in an analytic but also in a synthetic way.

5 Isometries of the hyperbolic plane

Isometries – mappings preserving the distance of all pairs of the mapped points. They are well-known even on basic school level and used in elementary synthetic constructions in Euclidean geometry.

In halfplane model, matrices of $PSL(2, \mathbb{R})$ form the group of all orientation preserving isometries. Hence, $PGL(2, \mathbb{R})$ are all isometries. These

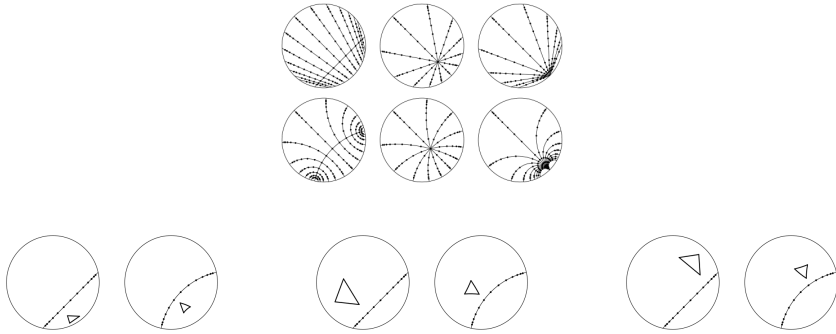


Fig. 5: Isometries (rotation upper row, reflection and glide reflection of a triangle, lower row) in the Klein and Poincaré model of hyperbolic plane. The neighboring points on the lines are uniformly distant.

are all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = \pm 1, a, b, c, d \in \mathbb{R}$$

with identification up to a multiple. Hyperbolic line reflections (line inversions, circle inversions) are generators of the group.

5.1 Classification of plane isometries

Various classification criteria can be used. Using minimum number of line reflections to chain, we get similarly to Euclidean plane identity, hyperbolic line reflection, hyperbolic translation, hyperbolic rotation (real point, ideal point), glide reflection (see fig. 5).

Interpreting the elements of $PGL(2, \mathbb{R})$ as Möbius transformation, $\frac{1}{z}$ represents circle inversion in unit circle centered at origin. In general, $f(z) = \frac{az+b}{cz+d}$ or $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ is a composition of translations, inversion, reflection in a real line, scaling and rotation for $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$.

Hyperbolic isometries can be represented with $a, b, c, d \in \mathbb{R}$. Geometry of complex projective line can be used in description of the hyperbolic plane. The loxodromic type of Möbius transformation cannot be found among the hyperbolic transformations.

5.2 Subgroups of hyperbolic isometries

The structure of subgroups of $PSL(2, \mathbb{R})$ is very rich. A subgroup Γ of isometries of discrete and free type corresponds in some sense uniquely to a hyperbolic surface of genus $g \geq 2$. Fairly heavy algebraic topology

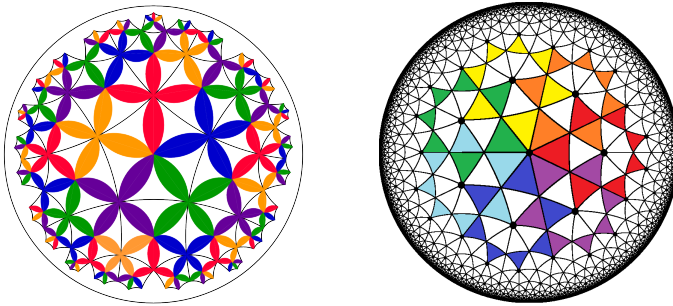


Fig. 6: Tiling of hyperbolic plane using regular 5-gon (left) and 7-gon (right).

and/or geometry theory is used for their classification. Such a group corresponds to certain tiling of the plane. There is a real $(6g+6)$ -dimensional Teichmüller space of non-isomorphic hyperbolic surfaces of genus g given as H^2/Γ .

Number theory studies modular group $PSL(2, \mathbb{Z})$ generated by transforms $z + 1$ and $-\frac{1}{z}$. Hence, many connections of hyperbolic geometry and number theory naturally appear.

A gateway to study 3-manifolds (see work of Thurston), their classification and further properties is the 2-dimensional hyperbolic plane. The higher dimensional approach is in many aspects similar. The higher dimensional transformas can be decomposed into a series of twodimensional transforms in a similar way to he Euclidean isometries. The results of 3-dimensional case are used physics for studying possible phenomena in real hyperbolic space around us.

6 Very few applications

6.1 Tilings and tesslations

The easiest tilings not to be found in the Euclidean setting is a tessellation by a regular n -gon for any $n \geq 3$ (see fig. 6). They can be found in the artworks of M. C. Escher. As mentioned above, the tilings can be produced using and appropriate subgroup of the isometries. They are called Fuchsian groups and their description requires some additional work. The algebraic/combinatorial/geometric properties are closely interconnected and explored.

Modern tessellations of surfaces in architecture can be a further extension in the setting of discrete differential geometry.

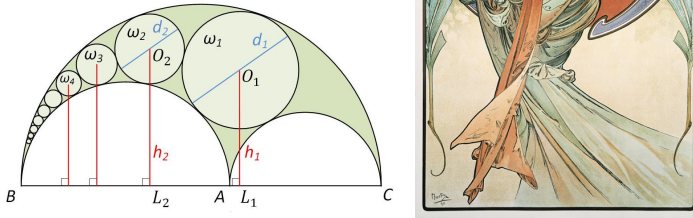


Fig. 7: Pappus Arbelos is a problem which can be easily transferred into a hyperbolic setting. It can be found in the art of Alfons Mucha.

6.2 Classical geometry problems

Classical problems of Euclidean geometry can be asked in the hyperbolic setting as well. Some of them have the same or almost same solution due to the fact that circles in hyperbolic setting can be identified with Euclidean circles within the Poincaré model. Lines there are also part of the circles. Hence Apollonian problems can be solved relatively easily. The metric problems are however much more difficult.

7 Conclusions

Hyperbolic geometry is a versatile area not only from the research point of view but also from the teaching point of view. Classical synthetic geometry can be used for explaining ideas, whereas analytical computations can be done in several ways – linear algebra, complex analysis and differential geometry approaches, even some geometric topology can be used.

Hyperbolic geometry is used in several areas of industry as a underlying space or as a tool. However, many classical problems with circles can be explored as problems of hyperbolic geometry or close areas such as Laguerre geometry.

Moreover the area generated many novel approaches in geometry, topology, combinatorics, e.g. the structure of the group of hyperbolic isometries is non-trivial and still many combinatorial as well as geometrical aspects can be studied.

The value teacher can bring is the synthetic understanding of many

time very technical approaches used in many different areas. Students often appreciate the interconnections and/or start to understand reasons why certain techniques are proper to use.

Acknowledgement

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Konvexní pětiúhelníky v matematických soutěžích

Convex pentagons in math competitions

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Abstract. The paper This is devoted to certain properties of the fundamental elements in convex pentagons — namely, the lengths of their sides and diagonals, the measures of their interior and exterior angles, and, where applicable, their area — as well as to their applications in solving specific planar geometry problems.

Keywords: convex pentagon, side length, diagonal length, size of interior and exterior angles.

Klíčová slova: Konvexní pětiúhelník, délka strany, délka úhlopříčky, velikost vnitřních a vnějších úhlů

1 Úvod

Ve srovnání s trojúhelníky a čtyřúhelníky se v naší matematické literatuře jen zřídka objevují příspěvky zaměřené na konvexní pětiúhelníky. Tento příspěvek se věnuje jejich základním vlastnostem – délkám stran a úhlopříček, velikostem vnitřních a vnějších úhlů, jeho obsahu – a jejich využití při řešení planimetrických úloh.

Úvodem si připomeňme větu o součtu velikostí vnitřních (i vnějších) úhlů v konvexním n -úhelníku.

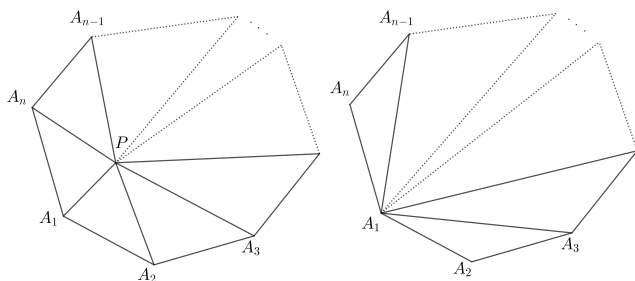
Věta 1

V každém konvexním n -úhelníku ($n \geq 3$) je součet velikostí jeho vnitřních úhlů roven $(n - 2) \cdot 180^\circ$.

Důkaz. Zvolme libovolný vnitřní bod P konvexního n -úhelníku $A_1 A_2 \dots A_n$, který spojíme se všemi jeho vrcholy (obr. 1 – vlevo). Součet velikostí vnitřních úhlů ve všech n takto vzniklých trojúhelnících (tzv. *triangulace* n -úhelníku $A_1 A_2 \dots A_n$) zmenšený o plný úhel, který je součtem velikostí vnitřních úhlů u vrcholu P ve všech trojúhelnících se společným vrcholem P , udává hledaný součet S velikostí všech vnitřních úhlů v tomto konvexním n -úhelníku. Platí tak

$$S = n \cdot 180^\circ - 360^\circ = n \cdot 180^\circ - 2 \cdot 180^\circ = (n - 2) \cdot 180^\circ,$$

což jsme chtěli dokázat.



Obr. 1: Věta 1

Jiný důkaz (viz např. v [4]). Zvolme libovolný z vrcholů uvažovaného n -úhelníku (bez újmy na obecnosti např. A_1), a pomocí všech jeho úhlopříček vycházejících z vrcholu A_1 jej rozdělíme na $n - 2$ trojúhelníků (obr. 1 – vpravo). Pro součet S velikostí všech vnitřních úhlů v konvexním n -úhelníku $A_1A_2 \dots A_n$ tak přímo obdržíme $S = (n - 2) \cdot 180^\circ$.

Důsledek 1

Součet velikostí všech vnitřních úhlů v každém konvexním pětiúhelníku je roven 540° .

Důsledek 2

V každém konvexním n -úhelníku ($n \geq 3$) je součet velikostí všech jeho vnějších úhlů roven 360° .

Důkaz. Podle výše uvedené věty je součet velikostí n vnějších úhlů v libovolném konvexním n -úhelníku roven rozdílu

$$n \cdot 180^\circ - (n - 2) \cdot 180^\circ = 2 \cdot 180^\circ = 360^\circ.$$

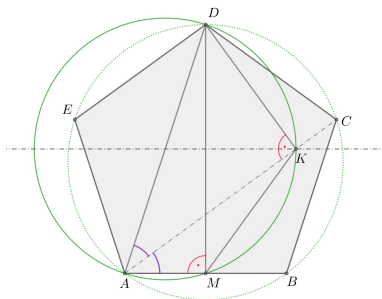
V další části uvedeme osm řešených úloh o konvexních pětiúhelnících.

Příklad 1

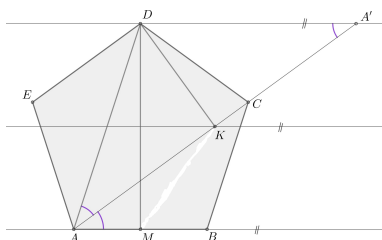
Je dán pravidelný pětiúhelník $ABCDE$. Označme M střed strany AB a K průsečík osy úsečky DM s úhlopříčkou AC . Dokažte, že $AK \perp DK$.

Řešení. Trojúhelník ADM je pravoúhlý s pravým úhlem při vrcholu M . Kružnice tomuto trojúhelníku opsaná je Thaletova kružnice nad přeponou AD . Bod K leží rovněž na této kružnici, jelikož je průsečíkem osy strany DM a osy vnitřního úhlu při vrcholu A . Tedy velikost úhlu AKD je 90° .

Jiné řešení. Uvažujme bod A' souměrně sdružený s vrcholem A vzhledem ke středu K . Protože bod K leží na ose úsečky DM , leží bod A' na



Obr. 2: Příklad 1



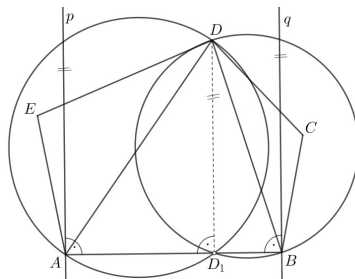
Obr. 3: Příklad 1 – jiné řešení

rovnoběžce s přímkou AB , která prochází vrcholem D (Obr. 3). Střídavé úhly $A'AB$ a $DA'A$ jsou shodné s úhlem DAC . Tedy trojúhelník $AA'D$ je rovnoramenný, přičemž bod K je středem jeho základny AA' . Odtud již přímo plyne $AK \perp DK$, což jsme chtěli dokázat.

Příklad 2

Je dán konvexní pětiúhelník, jehož všechny vnitřní úhly jsou tupé. Dokažte, že existuje taková dvojice jeho úhlopříček, že kruhy uvažované nad těmito úhlopříčkami (jako průměry) pokrývají daný pětiúhelník.

Řešení. K důkazu lze využít metodu extrémálního prvku. Bez újmy na obecnosti předpokládejme, že AB je nejdelší stranou konvexního pětiúhelníku $ABCDE$, který vyhovuje podmínkám úlohy. Kolmice k AB , které procházejí vrcholy A , B , označme po řadě p , q . Uvažujme nyní pás omezený rovnoběžkami p , q (obr. 3). Vrcholy C a E uvažovaného pětiúhelníku leží vně tohoto pásu, neboť úhly ABC a EAB jsou tupé. Současně však vrchol D tohoto pětiúhelníku leží uvnitř uvažovaného pásu (v opačném případě



Obr. 4: Příklad 2

by totiž strana AB nebyla nejdelší stranou pětiúhelníku $ABCDE$). Pata D_1 kolmice z vrcholu D k přímce AB je tedy vnitřním bodem strany AB uvažovaného pětiúhelníku.

Oba kruhy sestrojené nad průměry AD a BD tak evidentně pokrývají uvažovaný pětiúhelník $ABCDE$, což jsme chtěli dokázat.

Příklad 3

Je dán konvexní pětiúhelník $ABCDE$, v němž $|AB| = |BC| = |CD| = |DE|$, $|\angle ABC| = 96^\circ$ a $|\angle BCD| = |\angle CDE| = 108^\circ$. Určete, jakou velikost má jeho vnitřní úhel při vrcholu E .

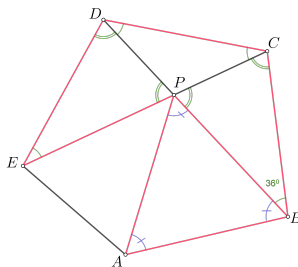
Řešení. Označme P průsečík úhlopříček BD a CE . Ze zadání plyne, že trojúhelníky BCD a CDE jsou rovnoamenné po řadě se základnami BD a CE . Přitom $|\angle CDB| = |\angle DBC| = |\angle ECD| = |\angle DEC| = 36^\circ$. Protože $|\angle BCD| = 108^\circ$, platí $|\angle BCP| = |\angle BPC| = 72^\circ$. Trojúhelník BCP je tedy rovnoramenný se základnou CP . Podobně zjistíme, že i trojúhelník DEP je rovnoramenný se základnou DP (obr. 5). Díky zadání platí také $|AB| = |BC| = |BP| = |EP|$ a dále $|\angle ABP| = 96^\circ - 36^\circ = 60^\circ$. Odtud již bezprostředně plyne, že trojúhelník ABP je rovnostranný a trojúhelník AEP je rovnoramenný se základnou AE . Dopočítáním vnitřních úhlů v rovnoramenném trojúhelníku APE snadno zjistíme, že $|\angle APE| = 180^\circ - 72^\circ - 60^\circ = 48^\circ$, tudíž $|\angle AEP| = 66^\circ$, a tedy $|\angle AED| = 36^\circ + 66^\circ = 102^\circ$. Velikost úhlu AED dopočítáme podle věty 1.

Příklad 4

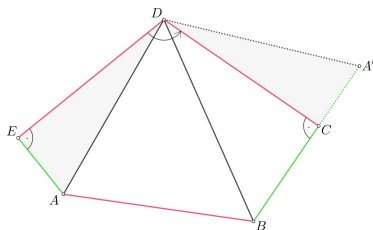
Je dán konvexní pětiúhelník $ABCDE$ s pravými úhly při vrcholech C a E , kde

$$|AB| = |CD| = |DE| = 1 \quad \text{a} \quad |BC| + |EA| = 1.$$

Dokažte, že jeho obsah je 1.



Obr. 5: Příklad 3



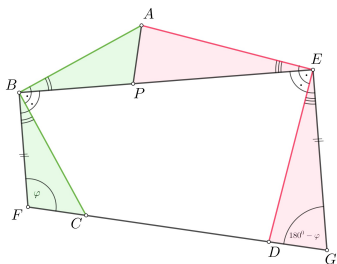
Obr. 6: Příklad 4

Řešení. V otočení se středem D a orientovaným úhlem EDC se vrchol E pětiúhelníku $ABCDE$ zobrazí na vrchol C , neboť podle zadání je $|DE| = |CD| = 1$. Pravoúhlý trojúhelník DEA s pravým úhlem při vrcholu E se v tomto otočení zobrazí na pravoúhlý trojúhelník DCA' , tj. platí $|EA| = |CA'|$, kde A' je obrazem vrcholu A . Bod A' pak leží na polopřímce BC za vrcholem C (obr. 6). Podle zadání platí $|BA'| = |BC| + |CA'| = |BC| + |EA| = 1$. Obsah pětiúhelníku $ABCDE$ je tak roven obsahu čtyřúhelníku $ABA'D$, v němž jsou trojúhelníky ABD , $A'BD$ shodné podle věty *sss*. Trojúhelník $A'BD$ má přitom obsah $1/2$, proto pětiúhelník $ABCDE$ má obsah 1, jak jsme chtěli dokázat.

Příklad 5

Je dán konvexní pětiúhelník $ABCDE$ s pravými úhly při vrcholech B a E , v němž platí $|AB| = |BC|$, $|DE| = |EA|$ a $|BE| = 10$ cm. Určete v cm^2 obsah daného pětiúhelníku $ABCDE$.

Řešení. Označme F , G průsečíky přímky CD s kolmicemi k přímce BE , které procházejí po řadě vrcholy B , E daného pětiúhelníku (obr. 7). Na



Obr. 7: Příklad 5

úhlopříčce BE uvažujme bod P , pro nějž platí $|BP| = |BF|$. V otočení se středem B a orientovaným úhlem -90° je přitom obrazem bodu P bod F a obrazem bodu A je bod C . Trojúhelníky ABP a CBF jsou tudíž podle věty *sus* shodné, neboť $|\angle ABP| = |\angle CBF|$ a také $|\angle APB| = |\angle CFB|$. Vzhledem k tomu, že přímky BF a EG jsou rovnoběžné, platí (s ohledem na shodnost trojúhelníků ABP a CBF)

$$|\angle DGE| = 180^\circ - |\angle CFB| = 180^\circ - |\angle APB| = |\angle APE|,$$

tedy trojúhelníky DGE a APE jsou také shodné (*usu*), a proto $|EG| = |EP|$. Navíc, první z nich je obrazem druhého v otočení se středem E a orientovaným úhlem $+90^\circ$. Dále platí $|BF| + |GE| = |BP| + |EP| = |BE| = 10 \text{ cm}$.

Obsah daného pětiúhelníku $ABCDE$ je tak roven obsahu lichoběžníku $BFGE$ se základnami BF a GE , tedy $\frac{1}{2} |BE|(|BF| + |GE|) = \frac{1}{2} |BE|^2 = 50 \text{ cm}^2$.

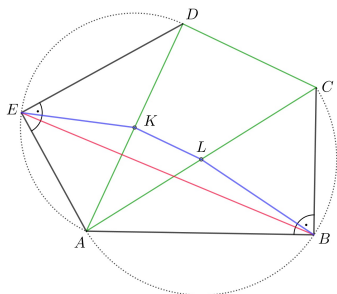
Příklad 6

Je dán konvexní pětiúhelník $ABCDE$ s pravými úhly při vrcholech B a E . Dokažte, že obvod trojúhelníku ACD není menší než $2|BE|$.

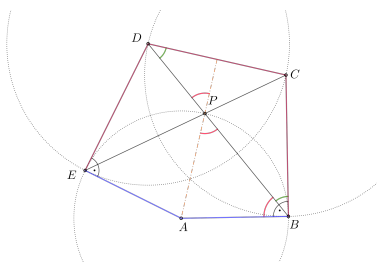
Řešení. Označme K, L středy úhlopříček po řadě AD, AC . Délka lomené čáry $BLKE$ je rovna polovině obvodu trojúhelníku ACD , protože platí: $|KL| = \frac{1}{2}|CD|$, $|EK| = \frac{1}{2}|AD|$ a $|BL| = \frac{1}{2}|AC|$. Tedy velikost úsečky BE je ostře menší než délka lomené čáry $BLKE$, která je rovna obvodu trojúhelníku ACD .

Příklad 7 (9. geometrická olympiáda I. F. Šarygina, 2013)

Je dán konvexní pětiúhelník $ABCDE$ s pravými úhly při vrcholech B a E , v němž $|AB| = |AE|$ a $|BC| = |CD| = |DE|$. Nechť P je průsečík jeho úhlopříček BD a CE . Dokažte, že $|PA| = |AB|$.



Obr. 8: Příklad 6



Obr. 9: Příklad 7

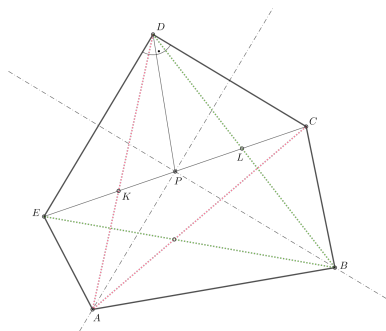
Řešení. Daný konvexní pětiúhelník $ABCDE$ je osově souměrný podle osy AP . Trojúhelník BCD je rovnoramenný se základnou BD a platí, že $|\angle DBC| = |\angle BDC|$. Dále je zřejmé, že, protože $|\angle ABP| + |\angle PBC| = \frac{1}{2}|\angle CPD| + |\angle PDC| = 90^\circ$. Odtud plyne, že $|\angle ABP| = |\angle APB|$, tedy trojúhelník ABP je rovnoramenný se základnou BP a $|PA| = |AB|$.

Příklad 8 (XVII. MO juniorů – Polsko, 2022, viz [5])

V konvexním pětiúhelníku $ABCDE$ s pravým úhlem při vrcholu D platí $|AC| = |AD|$ a $|BD| = |BE|$. Dokažte, že trojúhelník ABD a čtyřúhelník $ABCE$ mají stejný obsah.

Řešení. Označme K, L po řadě průsečíky úhlopříčky AD s EC a BD s EC . V lichoběžníku $APDE$ platí, že obsah trojúhelníku EAK se rovná obsahu trojúhelníku KPD . Analogicky v lichoběžníku $PBCD$ platí, že obsah trojúhelníku PLD se rovná obsahu trojúhelníku LBC . Tedy trojúhelník ABD a čtyřúhelník $ABCE$ mají stejný obsah.

Poznámka. Článek vznikl se souhlasem redakční rady časopisu MFI, neboť podstatná část článku již byla zde zveřejněna.



Obr. 10: Příklad 8

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Intersections of surfaces of revolution: Interactive modelling

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Abstract. In the Grasshopper programming environment, implemented within the Rhinoceros 3D modelling software, scripts generating CAD geometry in real-time can be created. The interactive nature of such scripts can serve as a tool for deepening the understanding of certain geometrical exercises presented during constructive geometry lectures. One such example is presented in this paper where a simple program capable of generating and displaying intersections of surfaces of revolution is made.

Keywords: Interactive modelling, surface of revolution, intersection, CAD

1 Introduction

In this paper, an approach to creating a simple Grasshopper [1] program for generating and visualizing the intersections of two surfaces of revolution [2][3] is presented. The Grasshopper programming environment is a part of the Rhinoceros 3D modelling software and allows for scripting in a visual node-based environment and for scripting using one of several implemented programming languages, for which libraries providing access to Grasshopper commands are provided. These commands allow for generating geometries from scratch, but also provide tools for manipulating geometry already present in the main Rhinoceros 3D viewport. One of the main features of Grasshopper is the real-time nature of the user interface, where changes to parameters used in the scripts affect the geometries, virtually, in real-time. The Grasshopper program presented in this paper shows how several input parameters and geometries can be combined to form a useful and customizable tool for use in lectures.

2 The program structure

As the goal is to create a program for modelling the intersection of two surfaces of revolution, several inputs governing the geometry to perform calculations on are needed. Defining the surfaces of revolution is the first step and can be done in many ways. Setting the generatrix curves and axes of rotation were chosen in this presented approach. As Grasshopper is capable of manipulating geometry created in the Rhinoceros viewport, the generatrix curves can be modelled directly within Rhinoceros and subsequently modified using the Grasshopper program. With this knowledge,

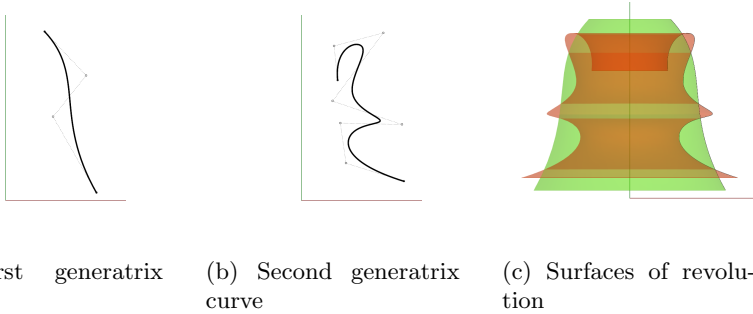


Fig. 1: User-defined generatrix curves and their respective surfaces of revolution

both generatrix curves can be modelled without the relative position of the two surfaces of revolution being taken into consideration yet, and their positioning can be carried out later on using the Grasshopper script. As the geometry can be altered later on in the Grasshopper program, explicitly defining the axes of revolution becomes unnecessary. To be able to gauge the scale of both surfaces of revolution in relation to each other, a reference axis of rotation becomes beneficial. Both generatrix curves may be modelled in respect to the same arbitrary axis of rotation, that can be defined, for example, as one of the world axes, thus not requiring any user input. In this approach, the world axis z was chosen for this exact function and both generatrix curves are to be modelled with respect to it. An example of two generatrix curves modelled in the Rhinoceros 3D viewport in the xz plane is shown in Figure 1.

These two generatrix curves are now the only two user input geometries required for defining the base surfaces used in the program. All of the other steps can now take place in the Grasshopper program, as the sequence of geometry modifications and other operations are going to be the same for all input geometries one were to define.

2.1 Grasshopper

Next step is to generate and position the surfaces of revolution governed by the two user-defined generatrix curves and the arbitrarily defined axes of rotation. This can be done in Grasshopper using any of the implemented programming language libraries or using the node-based system directly. In any case – a command for generating a surface of revolution is called and two input parameters are fed into it – a generatrix curve and an axis of rotation. Calling the command twice, once for each generatrix curve, supplying the world axis z as the axis of rotation both times, two surfaces of revolution are generated, as seen in Figure 1c.

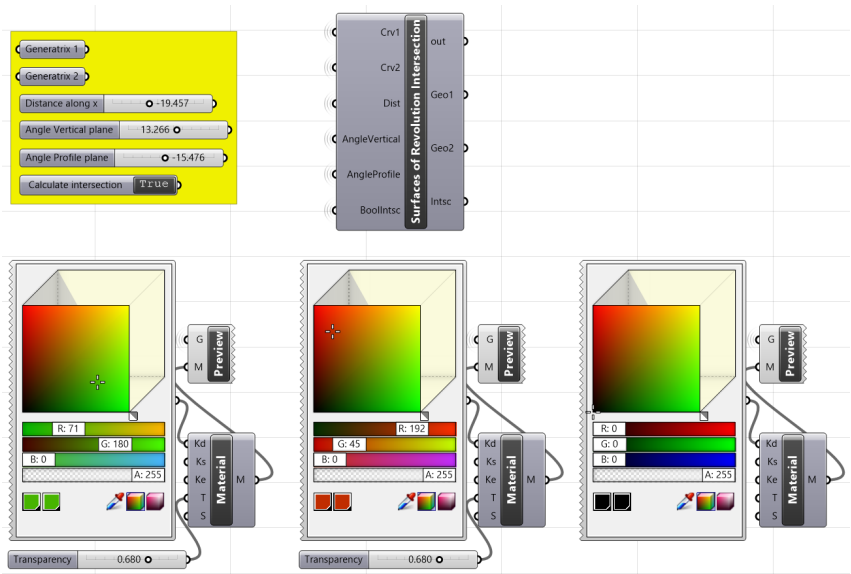


Fig. 2: Grasshopper user interface for the presented program

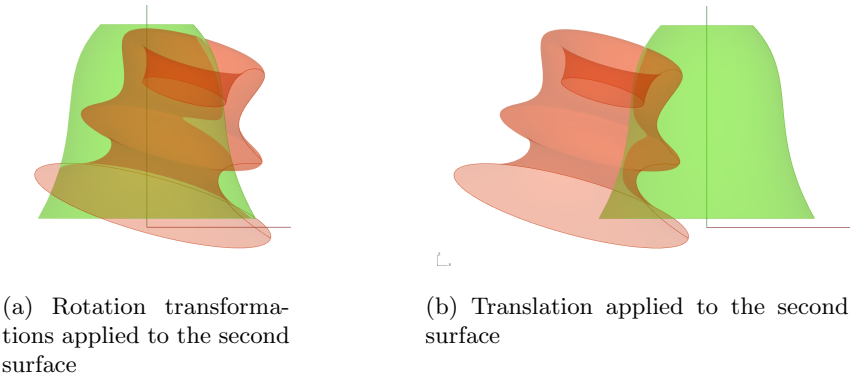


Fig. 3: Transformations applied to surfaces

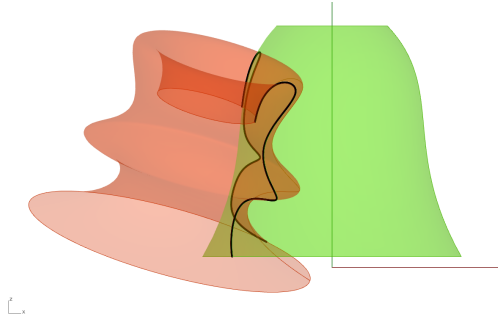


Fig. 4: Intersection curves of the two surfaces of revolution

Moving along in the program, next the relative position of the two surfaces of revolution should be defined. Transformations can be applied to the Grasshopper-generated geometry using a general Transform command. This command takes a 4x4 transformation matrix as an input. In order to set the relative position of the surfaces, one can remain stationary, and all of the desired transformations can be applied to the other, in order to make later adjustments more user-friendly. For the purposes of this paper, two rotations and one translation are going to be applied to one of the surfaces, and those will be all the degrees of freedom the user will have access to in order to define the relative position of the surfaces of revolution. The transformations will be applied to the red surface, generated from the second generatrix curve. First comes a rotation around the world y axis. Second is a rotation around the world x axis. The combined effect of these two affine transformations can be seen in Figure 3a.

The last transformation to apply is the translation. It was chosen to take place along the world x axis. Its effect can be seen in Figure 3b.

With all three of the chosen transformations applied, the last step is to calculate and display the surface intersection curves. Calling the surface intersection command calculates and returns all intersection curves present, also handling edge cases. The resultant display can be seen in Figure 4. In Listing 1 the code for the presented program is shown. The program was coded in the Grasshopper Python 3 module and uses the `rhinoscriptsyntax` library for geometry generation.

Listing 1: Python code of the program

```

1  """ Grasshopper - Surfaces of Revolution - Intersection - Script """
2  " Author: - Jakub - Kubat "
3  " Version: - 1.1 "
4
5  import rhinoscriptsyntax as rs
6  import math
7  Geo1 = []          # output list for the 1st surface of
                      # revolution and related geometries
8  Geo2 = []          # output list for the 2nd surface of
                      # revolution and related geometries
9  Intsc = []         # output list for the intersection curves
10
11 ##### FUNCTIONS #####
12
13 def Tx(x):          # transformation matrix for translation
                      # along global x
14     tx = [[1, 0, 0, x],
15           [0, 1, 0, 0],
16           [0, 0, 1, 0],
17           [0, 0, 0, 1]]
18     return tx
19
20 def Rx(a):          # transformation matrix for rotation
                      # around global x
21     c = math.cos(math.radians(a))
22     s = math.sin(math.radians(a))
23     R = [[1, 0, 0, 0],
24          [0, c, -s, 0],
25          [0, s, c, 0],
26          [0, 0, 0, 1]]
27     return R
28
29 def Ry(a):          # transformation matrix for rotation
                      # around global y
30     c = math.cos(math.radians(a))
31     s = math.sin(math.radians(a))
32     R = [[c, 0, -s, 0],
33          [0, 1, 0, 0],
34          [s, 0, c, 0],
35          [0, 0, 0, 1]]
36     return R
37
38 ##### SCRIPT #####
39
40 surf1 = rs.AddRevSrf(rs.coercecurve(Crv1), ((0, 0, 0), (0, 0, 1)))
                      # creation of 1st surface of revolution
41 surf2 = rs.AddRevSrf(rs.coercecurve(Crv2), ((0, 0, 0), (0, 0, 1)))
                      # creation of 2nd surface of revolution
42
43 rs.TransformObject(surf2, Rx(AngleVertical))          # rotation of
                      # 2nd surface of revolution around global x (angle from
                      # vertical plane)

```

```

44 rs.TransformObject(surf2,Ry(AngleProfile))      # rotation of
      2nd surface of revolution around global y (angle from
      profile plane)
45 rs.TransformObject(surf2,Tx(Dist))              # translation
      of 2nd surface of revolution along global x (distance
      between respective initial origins in the XY plane)
46
47 if BoolIntsc == True:    # INTERSECTION CALCULATIONS
48     Intersection = rs.IntersectBreps(rs.coercebrep(surf1),rs.
      coercebrep(surf2))    # intersection curves
49
50     if Intersection:    # appending intersection results to
      the proper output list
51         for i in range(len(Intersection)):
52             Intsc.append(rs.coercecurve(Intersection[i]))
53
54 Geo1.append(surf1) # appending 1st surface of revolution to
      the proper output list
55 Geo2.append(surf2) # appending 2nd surface of revolution to
      the proper output list

```

3 Conclusion

An approach to creating an interactive program in the Rhinoceros 3D Grasshopper environment was presented. Simple steps taken during the development of a teaching program were summarized, and the underlying decisions were discussed.

The Grasshopper environment provides ways to gather user input and use it directly as variable parameters in the programs created within. Some of the ways to input numerical variables in real time include widget-like nodes that remain visible when viewing the main Rhinoceros 3D viewport, allowing for parameter modifications to translate into visual changes directly, providing intuitive feedback and allowing for creative decisions to take place. This is the very reason Rhinoceros and Grasshopper were chosen to create this interactive intersection generator for use in lectures, because modifications to the generated geometry can be made immediately and can spark discussions between the students and the professor.

Acknowledgements

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Note on refinement curve length

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Abstract. A refinement curve is generated by repeatedly applying a refinement rule to a coarse control polygon. In our talk, we discuss the computation of the length of a refinement curve with a fixed number of iterations using Chaikin's algorithm. We also present several practical examples where the constant length of the refinement curve needs to be preserved.

Keywords: Subdivision, curve length, refinement.

1 Introduction

Subdivision curves are generated by iterative refinement of an input polyline [3, 5]. The limited amount of sources are concerned with the length of such curves [2], although the study of the length of curves in general is still an attractive topic [4]. In this work, we investigate the computation of the length of refinement curves obtained by Chaikin's scheme [1] after a fixed number of iterations. We also present computational details and experimental results for the construction of corner roundings, where the constant length of the refinement curve must be preserved.

2 Chaikin's Scheme

Let a polyline after k iterations be given by points $V_0^k, V_1^k, \dots, V_n^k \in \mathbb{R}^2$. Chaikin's refinement produces a polyline after $k+1$ iterations with points $V_0^{k+1}, V_1^{k+1}, \dots, V_{2n-1}^{k+1} \in \mathbb{R}^2$, defined by

$$V_{2i}^{k+1} = \frac{3}{4}V_i^k + \frac{1}{4}V_{i+1}^k, \quad V_{2i+1}^{k+1} = \frac{1}{4}V_i^k + \frac{3}{4}V_{i+1}^k.$$

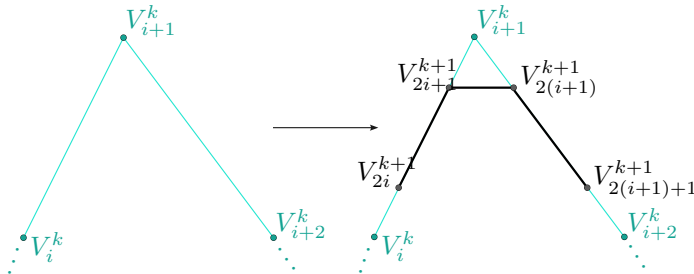


Fig. 1: Refinement step of Chaikin's scheme.

For a polyline $\mathcal{P}$ with three control points V_0^0, V_1^0, V_2^0 , the control points of the refined polyline after k iterations can be expressed in matrix form using the information from the previous iteration as

$$\begin{pmatrix} V_0^k \\ V_1^k \\ V_2^k \\ V_3^k \\ \vdots \\ V_{2^k}^k \\ V_{2^k+1}^k \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix} \begin{pmatrix} V_0^{k-1} \\ V_1^{k-1} \\ V_2^{k-1} \\ V_3^{k-1} \\ \vdots \\ V_{2^{k-1}}^{k-1} \\ V_{2^{k-1}+1}^{k-1} \end{pmatrix}$$

$$\mathbf{V}^k = \mathbf{S}^k \mathbf{V}^{k-1}, \quad \mathbf{S}^k \in \mathbb{R}^{(2^k+2) \times (2^{(k-1)}+2)}.$$

By recursively expanding, we obtain

$$\mathbf{V}^k = \mathbf{S}^k \mathbf{V}^{k-1} = \mathbf{S}^k \mathbf{S}^{k-1} \mathbf{V}^{k-2} = \cdots = \prod_{i=1}^k \mathbf{S}^i \mathbf{V}^0.$$

We can express the i -th row of the matrix $\prod_{i=1}^k \mathbf{S}^i$ as

$$\frac{1}{4^k} \begin{pmatrix} \vdots \\ a_i & (4^k - a_i - b_i) & b_i \\ \vdots \end{pmatrix},$$

where

$$a_i = \frac{(2^k - i + 2)(2^k - i + 1)}{2}, \quad b_i = \frac{(i - 1)(i - 2)}{2}.$$

Subsequently, we can compute the coordinates of the point V_i^k as the affine combination of input control points V_0^0, V_1^0 and V_2^0 as

$$V_i^k = \frac{a_i}{4^k} V_0^0 + \frac{4^k - a_i - b_i}{4^k} V_1^0 + \frac{b_i}{4^k} V_2^0.$$

2.1 Computation of Polyline Length

To compute the length ℓ of the polyline $V_0^k, \dots, V_{2^k+1}^k$ after k iterations of Chaikin's refinement of the polyline $\mathcal{P}$, we need to sum the lengths of the vectors given by two consecutive points $V_i^k, V_{i+1}^k, i = 0, \dots, 2^k$, i. e.

$$\ell = \sum_{i=0}^{2^k} |V_{i+1}^k - V_i^k|.$$

To express the length ℓ using only the input control points V_0^0, V_1^0 and V_2^0 , we use the matrix $\prod_{i=1}^k \mathbf{S}^i$. Denote by $\mathbf{B}^k$ and $\mathbf{T}^k$ the submatrices of $\prod_{i=1}^k \mathbf{S}^i$ obtained by removing the first and last row, respectively. Then the vectors $V_{i+1}^k - V_i^k$ may be expressed as

$$\begin{pmatrix} V_1^k - V_0^k \\ \vdots \\ V_{2^k+1}^k - V_{2^k}^k \end{pmatrix} = \mathbf{B}^k \begin{pmatrix} V_0^0 \\ V_1^0 \\ V_2^0 \end{pmatrix} - \mathbf{T}^k \begin{pmatrix} V_0^0 \\ V_1^0 \\ V_2^0 \end{pmatrix} = (\mathbf{B}^k - \mathbf{T}^k) \begin{pmatrix} V_0^0 \\ V_1^0 \\ V_2^0 \end{pmatrix}.$$

Since the vector $V_{i+1}^k - V_i^k$, $i = 0, \dots, 2^k$ may be computed as

$$V_{i+1}^k - V_i^k = \frac{-2^k + i}{4^k} V_0^0 + \frac{2^k - 2i}{4^k} V_1^0 + \frac{i}{4^k} V_2^0,$$

and if we denote the coordinates of the input coordinates $V_j^0 = (x_j, y_j)^\top$, $j = 0, 1, 2$, we can compute the length as

$$\ell = \frac{1}{4^k} \sum_{i=0}^{2^k} \sqrt{(\alpha_i x_0 + \beta_i x_1 + \gamma_i x_2)^2 + (\alpha_i y_0 + \beta_i y_1 + \gamma_i y_2)^2},$$

where

$$\alpha_i = -2^k + i, \quad \beta_i = 2^k - 2i, \quad \gamma_i = i.$$

3 Corner Rounding with Fixed Length

Consider an input polyline given by points V_0^0, V_1^0 and V_2^0 and fix the starting point of the rounding S on the segment $V_0^0 V_1^0$. Let $W_1 = V_1^0$ and let $W_0 = 2S - W_1$. Our task is to choose a point W_2 , so the refined curve after **fixed** number of iterations k with control points W_0, W_1, W_2 has **fixed** length ℓ .

All feasible points $W_2 = (x, y)^\top$ must satisfy the equation $g(x, y) = 0$, where

$$g(\mathbf{x}, \mathbf{y}) = \frac{1}{4^k} \sum_{i=0}^{2^k} \sqrt{(\alpha_i x_0 + \beta_i x_1 + \gamma_i \mathbf{x})^2 + (\alpha_i y_0 + \beta_i y_1 + \gamma_i \mathbf{y})^2} - \ell.$$

To determine the position of the point W_2 , we need to find the intersection of the ray $V_1^0 V_2^0$ with the curve $g(x, y) = 0$, see Fig. 2.

Computationally, it is more efficient to transform the input data into a coordinate system $\langle O', \mathbf{e}'_1, \mathbf{e}'_2 \rangle$, where

$$\begin{aligned} O' &= V_1^0 \\ \mathbf{e}'_2 &= \frac{V_2^0 - V_1^0}{\|V_2^0 - V_1^0\|} =: (x', y')^\top \\ \mathbf{e}'_1 &= (y', -x')^\top. \end{aligned}$$

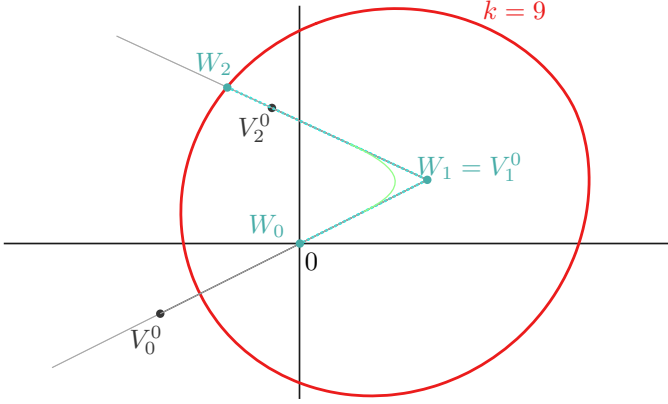


Fig. 2: Determination of the point W_2 on the curve $g(x, y) = 0$ for $k = 9$.

In this coordinate system, the x coordinate of the point W_2 vanishes. With $x = 0$, we may plug this value into the function $g(x, y)$ and we obtain the y coordinate as a root of the function

$$g_0(y) = g(0, y) = \frac{1}{4^k} \sum_{i=0}^{2^k} \sqrt{(\alpha_i x_0 + \beta_i x_1)^2 + (\alpha_i y_0 + \beta_i y_1 + \gamma_i y)^2} - \ell.$$

For the computation of the root we used the bisection method on the interval $[0, \infty)$. After the calculations, we transform the data back to the standard coordinate system $\langle O, e_1, e_2 \rangle$.

4 Examples

For illustration, consider the following cases. In the first example, we set the input polyline to $V_0^0 = (400, 200)^\top$, $V_1^0 = (100, 100)^\top$, $V_2^0 = (100, 400)^\top$. The number of iterations is set to $k = 9$, and we fix the length of the resulting curve to $\ell = 300$. The starting point of the rounding S is defined to be on the segment $V_0^0 V_1^0$, i. e. $S = (1 - t)V_0^0 + tV_1^0$, where $t \in [0, 1]$. As we see in Fig. 3, various placements of the starting point S produce a correct rounding for the input polyline and for the increasing values of the parameter t , the end point of the rounding approaches the last input point V_2^0 . As a consequence, for large values of t , the rounding may exceed the input polyline.

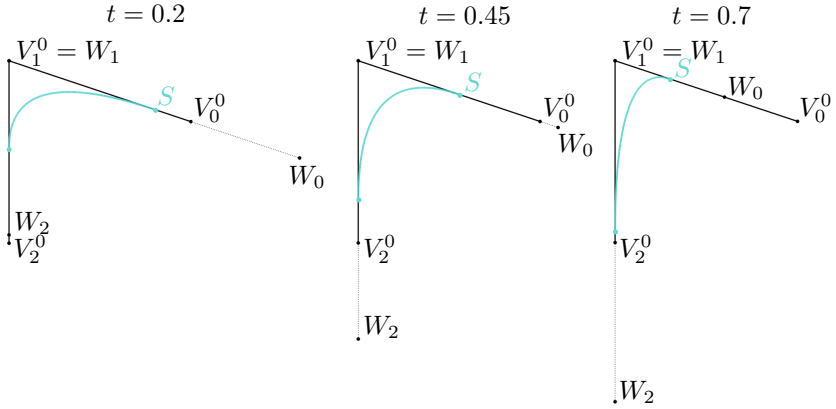


Fig. 3: Corner rounding with fixed length $\ell = 300$ with various placements of the starting point S .

In the second example, the input polyline is given by $V_0^0 = (200, 100)^\top$, $V_1^0 = (400, 300)^\top$, $V_2^0 = (100, 600)^\top$. Again, we set the number of interactions to $k = 9$, as seen in Fig. 4. However, now we fix the starting point as the midpoint $S = 0.5V_0^0 + 0.5V_1^0$ and inspect various lengths of the refined curve. Similarly as in the previous example, the endpoint of the rounding approaches the last input point with increasing length and may overshoot it for sufficiently large lengths.

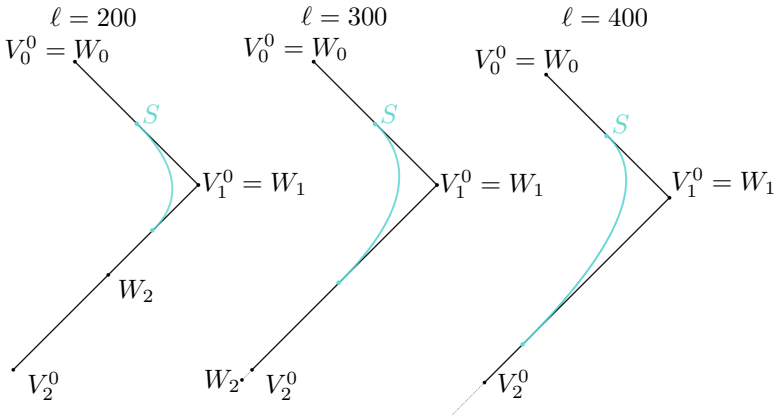


Fig. 4: Corner rounding with various lengths.

5 Conclusion

We have presented a method for computing the length of the refinement curve obtained by k iterations of Chaikin's rule. A formula was provided that avoids multiplication of large matrices. Furthermore, we demonstrated how to construct a refinement curve of fixed length by imposing additional constraints. Future work includes formalizing the constraints to ensure that the curve does not exceed the input control polygon, generalizing the method to control polygons with an arbitrary number of points, and exploring different constraints and subdivision rules to generate refinement curves of fixed length.

Acknowledgements

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A notelet to gyroid: Straight lines invited

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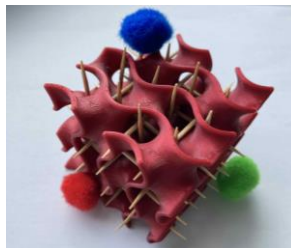
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Abstract. A triply periodic surface gyroid and 3D printing technology are, roughly speaking, contemporaries. Technology experts share an opinion on impossibility to produce physical model of gyroid before 3D printing era. An commonly known aesthetical attractiveness as well as wide technical applicabilities spectrum of gyroid are enriched here by our original view on mutual position of straight lines and gyroid. Appendices offer some alphabet-like items on gyroid and a many different experiences with it. E. g. involving gyroid-phenomenon to elementary course context of mathematics at technical university is educational offer. Also challenges to meditate on correlation input-output algorithmical hierarchy if starting point is verbal prompt, for ChatGPT, to sketch of an abandoned large 3D printed gyroid model in the university yard and output looks like ... is it gyroid? See inside a contribution. And others.



Keywords: gyroid, surface, 3D print, straight line, graphic calculator, ChatGPT, sustainability, metamaterial, Gaussian bell-curve

1 Introduction

The gyroid is a surface. Triply periodic one. It is aesthetically magical on the one hand. It is superstar in many art galleries. But it has also various uses from point of view of mathematics and/or physics. Gyroid can serve as

- an geometrical icon of “open” structures,
- representant of lightweight materials,
- inspiration for metamaterials creating – thanks to existence of its mathematical representation via implicit equation of three variables in 3D orthogonal coordinate system,

- useful and beautiful infill in 3D printing; there are some “odes on gyroid”, e. g. in [1], as follows, quoted: “gyroid is hands down the best. as a plus, it looks good on timelapses.”, “I do use gyroid on all translucent prints though. Can’t beat that cool infill pattern showing through.”, “Gyroid is def the strongest, I pretty much use it in 90% of my prints.”

- motivation for architectonic structures [2],
- alternative for al dente-pasta-like in nuclear stars simulations [3, 4, 5, 6], etc.

In this paper we offer simple as well as original view on a gyroid in combination with parallel straight lines placed to series of gyroidal holes. We combine representation of a scene via a) digital simulation, b) photo of physical model, c) mathematical equations.

There is plenty of rational attitudes to gyroid. They agree with Gauss-bell curve distribution. There are gyroid-bestoids fans, also gyroid-fans – the enemies of lovers of straight lines.

2 Method

An equation of gyroid in implicit form, in 3D Cartesian coordinate system ($O; x, y, z$), looks like $\sin x \cos y + \sin y \cos z + \sin z \cos x = 0$.

By observing the physical model of the gyroid and straight lines/wooden skewers, Fig. 1, and with the help of digital visualization of the scene [7], Fig. 2, we came to the following conclusions:

A) the parallel straight lines of one set are skewed (SK: mimobežné) with ones in next two sets of straight lines; lines of one set are perpendicular to the cube wall.

B) Four sets of parallel straight lines in one set: lines inside one set are of body diagonal directions, correspondingly. For positive real number c , these are yellow (c, c, c) , red $(-c, c, c)$, green $(c, -c, c)$ and blue $(c, c, -c)$ direction vectors, Fig. 3.

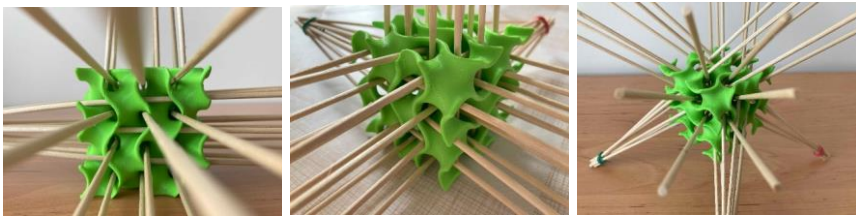


Fig. 1: Gyroid and straight lines/wooden skewers. Sticks of the same direction, at the right figure, are for the practical simplicity pulled together with a rubber band.

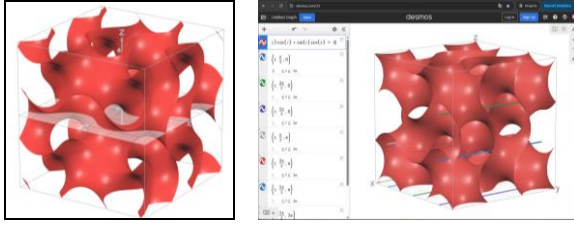
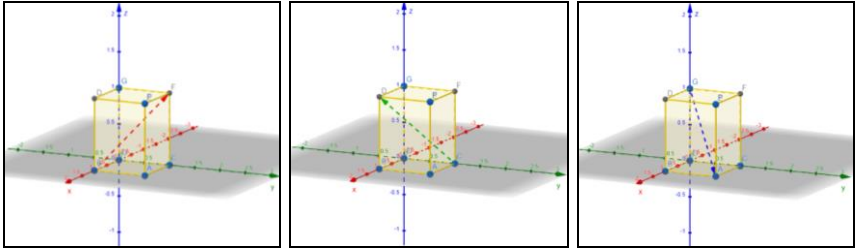


Fig. 2: Digital visualization of the scene, Desmos; gyroid alone (left) an gyroid with set of parallel lines (right).



$$\mathbf{r} = (-c, c, c)$$

$$\mathbf{g} = (c, -c, c)$$

$$\mathbf{b} = (c, c, -c)$$

Fig. 3: Direction vectors for $c = 1$; yellow one, $\mathbf{y} = (c, c, c)$, is not figured here.

3 Discussion

Although not very deep considerations about the gyroid and lines, nevertheless they have led to the generation of a rich palette of questions, opportunities, challenges, e. g.:

- multilevel pedestrian communications inside gyroid-shaped-structures.
- cylindrical helixes with appropriate radius instead of straight lines, with constant/nonconstant thread height,
- instead of straight lines, take e.g. straight cylindrical surfaces (with a defining line, a circle, etc., but also with some non-closed defining line) or some suitable translational surfaces and let them pass through a series of gyroid holes,
- instead of lines/surfaces, take a string and explore the properties of threading it through the holes of a gyroid,
- field of acoustics: sound amplification by gyroid, diffraction, interference,
-
- flow of media (liquid, air/gas, light) through a gyroid; could it be used as a splitter? That is, (for example) light coming from somewhere would be split into several directions when it exits,

- intersection of a gyroid with spherical/other – non-planar surfaces (with a common center (at a gyroid point or in a hole), different R or with mutually displaced centers and the same R).

4 Conclusion

Suppose we have gyroidal structure placed in cube. We present here figures on digital simulation, photos and info on skewness (in Slovak: mimobežnosť) of the straight lines, where they go through the series of holes, perpendicularly on the cube walls. Similarly, scene with straight lines of the some direction as a body diagonals have. In this second case lines have an intersection.

Acknowledgements

This contribution was supported by the Slovak Cultural and Educational Grant Agency (KEGA) under Project No. 009STU-4/2023. The authors are especially grateful to our colleague DM. He and coauthor RĎ were able to make a 3D printed Gulliver gyroid. Also Kevin's gyroid from Dubnica is also successfully fulfilling its educational and awareness mission. Special thanks go to our colleagues and conference participants who contributed to our understanding of gyroids with their knowledge and experience.

Personal note from IM: this notelet has its roots in the enlightenment and education of the high-energy physicist Ján Weiss and the unique Slovak geometer Jozef Zámožík.

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Appendices

Appendix B Beauty. Gyroid – not only Bestoid. Is is also Beautoid, Fig. 4.



Fig. 4: a) Gyroid – Beautoid. Gyroid and its shadows in sunshine, b) Gyroid geometry offers a multitude of possibilities for tactile sensations.

Appendix C ChatGPT. Challenges to meditate on correlation input-output algorithmical hierarchy if starting point is verbal prompt, for ChatGPT, to sketch of an abandoned large 3D printed gyroid model in the university yard and output looks like ... see left figure 5a. Is this gyroid? Let the loneliness of the gyroid deepen, and time leaves a strong mark on its appearance, Fig. 5b. When we put ChatGPT Abraham's-like-prompt on about "at least one man" with a sense of responsibility, the AI's graphical response was enhanced with a water jug and a wooden sign with the warning "Do Not Remove", Fig. 5c.



Fig. 5: a) An abandoned large 3D printed gyroid, b) time leaves a strong mark on its appearance, c) added water jug and a wooden sign with the warning "Do Not Remove". ChatGPT Images, Aug 8, 2025.

Comparison of i) ChatGPT-created gyroid, ii) one generated via graphical calculator and finally iii) real 3D printed object (Fig. 6) takes together these findings, these compatibilities vs. incompatibilities:

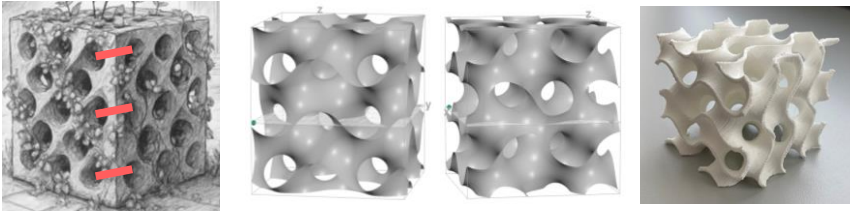


Fig. 6: a) Is the first, AI-generated object gyroid? Compare it with b) output from graphical calculator and with c) real 3D printed gyroid on photo.

Object on a) is open structure, it has triply periodicity – like the gyroid. Some kind of twining plant is given but it is not an ivy of common shape. Discrepancies are in incorrect position of holes in two neighbour cube faces, long and short paths with no-zero thickness, in 3D print, absent; pronounced, distinct vertices and edges are not present in real gyroid; finally, the first object looks more like solid with holes then like a surface.

Appendix GG Gulliver gyroid. October 3, 2025 is a birthday of Gulliver gyroid, fig 7. Our colleague DM, with the help of RĎ from authors team, finished making a beautiful, relatively large gyroid, with an edge length of 30 cm. Gulliver consists of five layers. Time of 3D printing of one layer was about the whole 24 hours. So far, approximately one hundred people – some of our students in Trnava and Dubnica nad Váhom, colleagues and coauthor IM's family members have been able to meet the Gulliver gyroid.



Fig. 7: The Gulliver gyroid in various situations

Appendix E Education. One of central tips for involving gyroid-phenomenon to elementary course context of mathematics at a technical university is visualisation of gyroid in some of common graphical web calculators [7]. It is a surface, but it is not a graph of a function of several variables. In this way, the

gyroid can reinforce the knowledge of the difference between these two types of graphs. Similarly, HS and IM from authors team found, that handling with planar sections of gyroid, for variability of plane orientation, Fig. 8, can serve as the next way of useful training of spatial imagination.

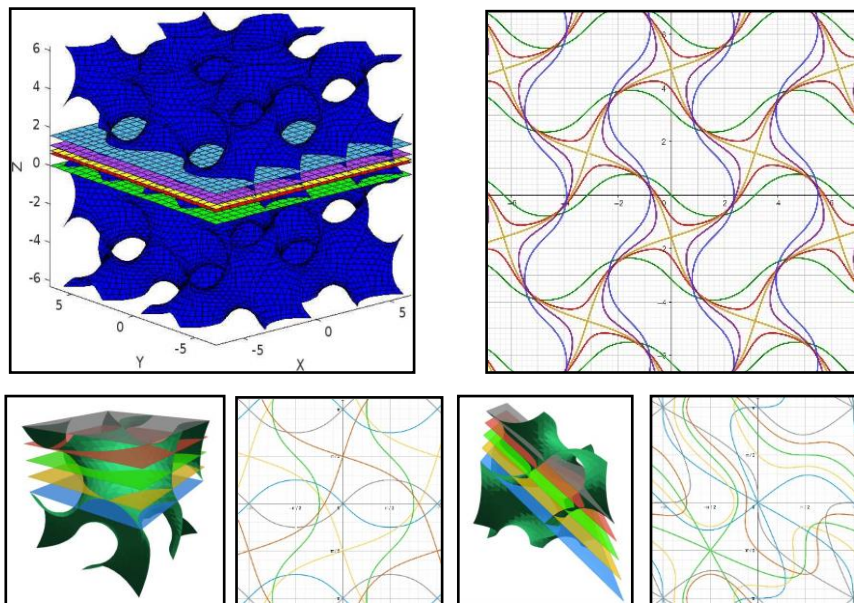


Fig. 8: Three pairs of various planes orientations and positions, and their intersections with gyroid, correspondingly.

Below, on Fig. 9, are four student's quick sketches of gyroid, within a few minutes. Figs are significantly different, but all have in themselves something "gyroidal".

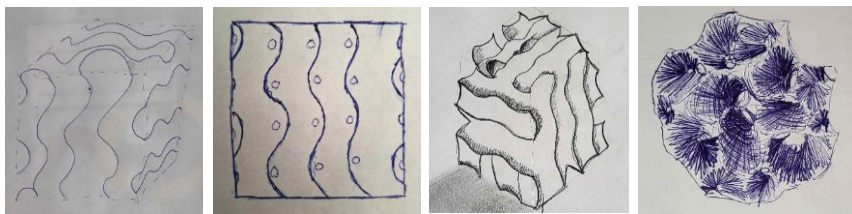


Fig. 9: Students' sketches of gyroid, Secondary Industrial School of Transport in Trnava, June 2025

Here are alternatives for involving gyroid-phenomenon into elementary course context of mathematics at technical university/secondary school:

- trig functions, • curve/line – surface – solid/massiv, • functions of one variable or more variables, • function vs. non-function, • explicit or implicit or parametric way of object description, • orthogonal projections of something, • mutual position of ..., • transformations of identity/similarity, • linear or nonlinear transformations, • handling with variables/functions, etc.

Appendix P Powder. There are many types of 3D printing, e. g. from filament, from powder, where special safety attention is required, etc. Flexible powder-based gyroid (Fig. 10) was ensured thanks to coauthor RB.



Fig. 10: Flexible powder-based, 3D printed gyroid.

Appendix S Spontaneity. It is impossible not to add a spontaneous cry of amazement at the gyroid of my colleague-friend MM, 2025-07-29: „His eyes! Jumping nut! I thought it would be easier! :D” (In Slovak: “Jeho oči! Matička skákavá! Myslela som si, že to bude dačo jednoduchšie! :D“)

Appendix T Trypophobia. It would probably be strange if there was no other phenomenon, next view of the gyroid than fascination and respect. A Gaussian bell curve applies. As IM' colleague PŠ informed, there is a trypophobia in nature. Maybe, it can be taken as undesirable, although statistically natural, counterbalance to aesthetic and practical enthusiasm for gyroid.

On Kármány type triangle geometry by geometric (Grassmann-Clifford) algebra

To memory of my professor and doctor father
Ferenc Kármány (1907-1989)

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Abstract. To memory of Ferenc Kármány I follow his didactical credo. Draw triangles outward on sides of a given triangle ABC in the Euclidean plane, say $AB\bar{C}$, $BC\bar{A}$, $CA\bar{B}$, and consider the segments $A\bar{A}$, $B\bar{B}$, $C\bar{C}$. Special starting conditions guarantee that the above segments intersect each other in a point K , called Kármány points to his honour. The later (~1850) strong and very useful machinery (by Grassmann and Clifford, in analogy of vector cross product) serves us a unified method and further interesting discussions.

Keywords: Triangle geometry, projective geometric (Grassmann-Clifford) algebra, absolute geometry by János Bolyai, problem solving by geometric transformation



F. KÁRMÁNY

retired professor of the L. Eötvös University,
member of the

Editorial Board of the Annales Univ. Sci. Budapest, Sec. Math.
died on May 9, 1989.

Let us recall a **well-known school task**: *In the (Euclidean E^2) plane of a triangle ABC we draw regular triangles outward on sides of ABC , say $AB\bar{C}$, $BC\bar{A}$, $CA\bar{B}$, respectively. Prove that the segments $A\bar{A}$, $B\bar{B}$, $C\bar{C}$ intersect each other in a point K , that is the isogonal point of ABC and the distance sum $AK + BK + CK$ is minimal for K among all points of the plane (Fig. 1).*

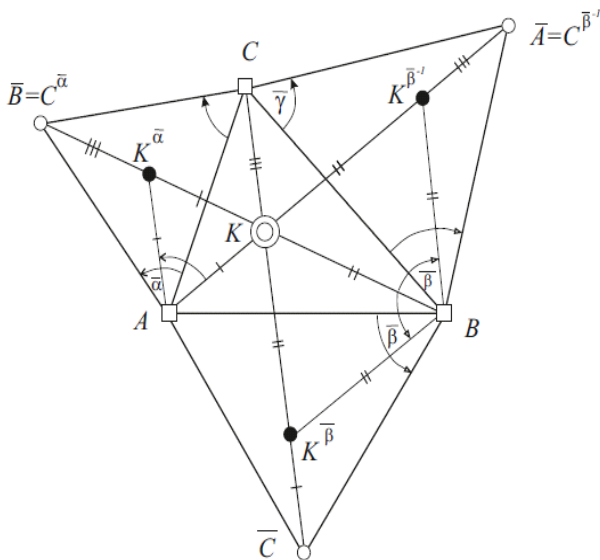


Fig. 1: K is the isogonal (or Fermat-Torricelli) point of triangle ABC with minimal distance sum $KA + KB + KC = A\bar{A} = B\bar{B} = C\bar{C}$.

Professor Kárteszi noticed that instead of regular triangles we can draw isosceles ones with all equal base angles, and the above K (called *Kárteszi point*) exists also in the Bolyai–Lobachevsky hyperbolic plane H^2 (in the sphere S^2 as well, (see also Kálmán, 1989 and Sect. 2), the orthocentre, barycentre are specific cases.

This problem leads also to a more general Yaglom triangle configuration and to an analogous extremal point K , where the distance sum $\alpha KA + \beta KB + \gamma KC$ is minimal with given positive real numbers α, β, γ (think of $\alpha = 4, \beta = 3, \gamma = 2$, Figs. 2, 3.).

Moreover, as a new result of our previous paper, an extension onto "absolute plane" (S^2, E^2, H^2 , Minkowski plane M^2 , Galilei (or isotropic) plane G^2) can be formulated and solved by three reflections theorem (see e.g. Bachman [1], Molnár, 1978, [2], and Weiss, 2018, [13]), and geometric (Grassmann–Clifford type) algebra (Perwass et al., 2004 and Sect. 3, [12]).

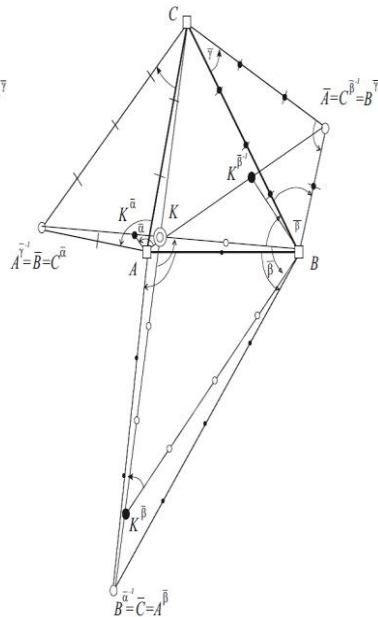
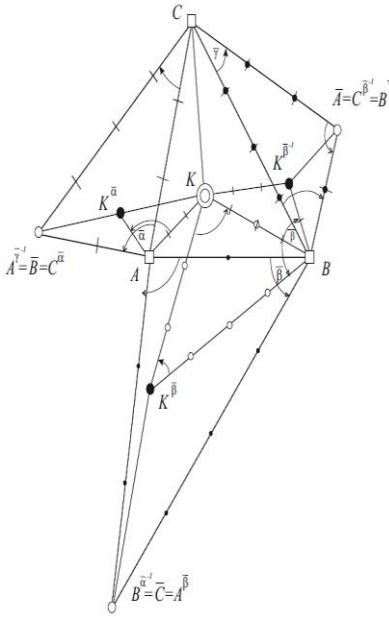


Fig. 2: Segmentation (then linearization) of $\alpha KA + \beta KB + \gamma KC$ (say $\alpha = 4, \beta = 3, \gamma = 2$) by rotatory similarities $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ about A, B, C , with angles $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, respectively (sketch).

Fig. 3: The construction of the extremal point $K = \bar{A}\bar{A} \cap \bar{B}\bar{B} \cap \bar{C}\bar{C}$ to Yaglom's problem in Fig. 2. Here $\alpha : \beta : \gamma \sim 4 : 3 : 2$ (satisfy the triangle inequalities!), and $\alpha \bar{A}\bar{A} = \beta \bar{B}\bar{B} = \gamma \bar{C}\bar{C}$.

Open problems arise as well. By this we want to follow F. Kármányi's didactical credo (see also his wonderful book Kármányi 1976, [5] of great international success):

Start with a natural, elementary, visually well understandable task! Then follow the manipulations, tools, new mathematical concepts, the technical machinery; then the solution, occasional theory, further applications, extensions ...

In the presentation and future papers the projective geometric (Grassmann-Clifford type) algebra leads us to the "most general" extension of Kármányi point K to triangle ABC with outward drawn triangles $AB\bar{C}, BC\bar{A}, CA\bar{B}$ and $K = \bar{A}\bar{A} \cap \bar{B}\bar{B} \cap \bar{C}\bar{C}$.

Theorem. For the existence of Kárteszi point it holds the criterion (if and only if; now $*$ means the multiplication point):

$$\sin \alpha_{\bar{b}} / \sin(\alpha + \alpha_{\bar{b}}) * \sin \beta_{\bar{c}} / \sin(\beta + \beta_{\bar{c}}) * \sin \gamma_{\bar{a}} / \sin(\gamma + \gamma_{\bar{a}}) = \\ \sin \alpha_{\bar{c}} / \sin(\alpha + \alpha_{\bar{c}}) * \sin \beta_{\bar{a}} / \sin(\beta + \beta_{\bar{a}}) * \sin \gamma_{\bar{b}} / \sin(\gamma + \gamma_{\bar{b}})$$

Here e.g. $\alpha_{\bar{b}}$ denotes the outward angle at vertex A on the side b , and $-$ for simplicity – every angle argument falls between 0 and π . Thus K will be in the interior of triangle ABC , as most important situation. (This is a transcendent equation, in general!)

In references [6], [11] we recall Ferenc Kárteszi as a teacher and scientist personality.

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Lie group approach to envelope surfaces

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Abstract. The construction and analysis of envelope surfaces are of great interest from both theoretical and practical perspectives. Certain special cases, such as the envelopes of (truncated) cones undergoing Euclidean motion, are particularly relevant for applications in CNC manufacturing. We describe 1-parameter systems of surfaces as curves in the homogeneous spaces of appropriate Lie groups. Using the Lie group formalism, we rigorously express the inherent symmetry and linearity in the computation of the envelope.

Keywords: envelope surface, characteristic curve, Lie group, homogeneous space, tangent mapping.

1 Introduction

The topic of this paper is the connection between envelope surfaces and Lie groups. Lie groups and their homogeneous spaces are two of the most widely used mathematical concepts, see e.g. [1, 2]. They are useful for expressing symmetries. Envelope surfaces have been well studied for both their theoretical aspects and their applications, with some special cases being studied in geometric modelling, including canal surfaces [3] and developable surfaces [4]. The envelopes of moving cones are especially useful for applications in CNC manufacturing.

This paper ties the two concepts together. We apply a Lie group formalism to the computation of envelopes, in an attempt to exploit the symmetries hidden in our system and simplify the computation. The section 2 sums up the theory necessary for our approach. Then, in section 3 we showcase some novel examples, that provide insight into the computation method. Finally, we conclude the paper.

2 Theory

Let us briefly summarize the necessary concepts from Lie theory and envelope computation, before tying them together. The resulting computational methods will follow closely the theory developed in [5], thus the proofs will be omitted from here and can be found in the aforementioned paper.

For the sake of this paper, let us consider an implicitly defined object F embedded in $\mathbb{R}^n$ with its implicit equation f . That is

$$F = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0\}.$$

We will refer to F as an *elementary object*. Typically, we will consider F to be a surface in $\mathbb{R}^3$ (e. g. a plane $F = \{[x, y, z] \in \mathbb{R}^3 \mid z = 0\}$) or a curve in $\mathbb{R}^2$ (e. g. a parabola $F = \{[x, y] \in \mathbb{R}^2 \mid x^2 - y + 1 = 0\}$).

A real smooth manifold $\mathcal{G}$ of dimension d with a binary operation $\circ$ is called a *Lie group*, if

- $(\mathcal{G}, \circ)$ is a group,
- the mappings $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} : (x, y) \rightarrow x \circ y$ and $\mathcal{G} \rightarrow \mathcal{G} : x \rightarrow x^{-1}$ are smooth.

The tangent space to $\mathcal{G}$ at its unit element $T_1\mathcal{G}$ is called the *associated Lie algebra* $\mathfrak{g}$. See [1, 2] for more thorough introduction.

Example: As the most important example, let us consider the group of direct isometries of a Euclidean space, also referred to as *special Euclidean group*. In $\mathbb{R}^2$ the group is denoted $\text{SE}(2)$ and has dimension 3, in $\mathbb{R}^3$ the group is denoted $\text{SE}(3)$ and has dimension 6.

All direct isometries can be expressed as $A\mathbf{x} + \mathbf{b}$, where A is a rotation matrix satisfying $AA^T = I$ and $\det A = 1$, while $\mathbf{b}$ is a translation vector. The group can be embedded into the group of $(n+1) \times (n+1)$ matrices via

$$(A, \mathbf{b}) \mapsto \begin{pmatrix} A & \mathbf{b} \\ 0 \dots 0 & 1 \end{pmatrix},$$

with isometry mapping

$$\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} A & \mathbf{b} \\ 0 \dots 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}.$$

In our case, we require the group $\mathcal{G}$ to act smoothly on $\mathbb{R}^n$. This allows us to define a 1-parametric transformation $g_t \subseteq \mathcal{G}$ for t in some interval I , which can be applied to our elementary object F , thus defining a 1-parametric system

$$\mathcal{F} = \{g_t(F) \mid t \in I\}.$$

Note, that the elementary object F does not have to lie in $\mathcal{F}$. Also, observe that each object $F_t = g_t(F)$ in this system can be also defined by implicitly as

$$F_t = \{\mathbf{x} \in \mathbb{R}^n \mid f_t(\mathbf{x}) = 0\}, \text{ where } f_t(\mathbf{x}) = f(g_t^{-1}(\mathbf{x})).$$

Now, we would like to compute the envelope of our system F_t . Using the characterization by [6, 7], we may define *characteristic sets* as

$$\chi_t = \{\mathbf{x} \in \mathbb{R}^n \mid f_t(\mathbf{x}) = 0 \wedge \frac{\partial f_t}{\partial t}(\mathbf{x}) = 0\}.$$

The *envelope* χ of our system will then be the union of these characteristic sets, i. e.

$$\chi = \bigcup_{t \in I} \chi_t.$$

Finally, let us define the *linear space of functions* Q as follows

$$Q = \text{span}\{q: \mathbb{R}^n \rightarrow \mathbb{R} \mid q(\mathbf{x}) = f(g^{-1}(\mathbf{x})), g \in \mathcal{G}\}.$$

Note, that Q contains all implicit equations f_t of the system F_t , as well as $f = f \circ \text{id}$. For the sake of notation, we also define a map $\phi: \mathcal{G} \rightarrow Q$, that to each element $g \in \mathcal{G}$ assigns the corresponding function $\phi(g) = f \circ g^{-1}$.

Theorem: Let F be an elementary object in $\mathbb{R}^n$ with implicit equation f and $\mathcal{G}$ a Lie group containing 1-parameter system g_t for t in some interval I . Then there exists a *preimage set* $\bar{\chi}_t$ such that

$$\chi_t = g_t(\bar{\chi}_t).$$

Moreover, for any $t_0 \in I$, there exists $\gamma_{t_0} \in \mathfrak{g}$ such that

$$\bar{\chi}_{t_0} = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0 \wedge d_{\text{id}}\phi(\gamma_{t_0})(\mathbf{x}) = 0\},$$

where $d_{\text{id}}\phi: \mathfrak{g} \rightarrow Q$ is a tangent mapping at identity element of our Lie group $\mathcal{G}$.

The theorem for $n = 3$ is just a proposition 11 from [5]. Since the proof in the paper does not depend on dimension, it also applies to this more general case.

2.1 Finding the envelope

The computation method is as follows. We are given an elementary object, defined by its implicit equation $f(\mathbf{x}) = 0$, and the corresponding transformation $g_t \subseteq \mathcal{G}$ which is part of a suitable Lie group.

Now, we want to compute the preimage sets $\bar{\chi}_t$. For this, we need not only the elementary object $f(\mathbf{x}) = 0$, but also the equation $d_{\text{id}}\phi(\gamma_{t_0})(\mathbf{x}) = 0$. To make use of the linearity of tangent mapping, we will need to evaluate it on the basis of Lie algebra $\mathfrak{g}$ associated to Lie group $\mathcal{G}$.

Here we need the transformations $g(t)$ that generate our Lie group $\mathcal{G}$. For example, in standard Euclidean group these will be the rotations and translations. Differentiating $g(t)$ with respect to t will yield a basis element γ of the associated Lie algebra $\mathfrak{g}$. Differentiating $\phi(g(t))$ with respect to t will yield the desired element $d_{\text{id}}\phi(\gamma)$, as this is the definition of tangent mapping.

To obtain the element γ_{t_0} , we will use the proof of proposition 11 in [5]. Hence, we first define $h(t) := g_{t_0}^{-1}g_t$ and set $\gamma_{t_0} = h'(t_0)$. Once we

express γ_{t_0} in our basis, we are able to use the computed coefficients to write out the desired equation.

The examples in following section will hopefully make the method more clear.

3 Examples

Let us now illustrate the procedure on couple of examples. We will proceed with the case of moving parabola in $\mathbb{R}^2$, then move on to example in $\mathbb{R}^3$.

3.1 Moving parabola

Let us consider a parabola given by

$$x^2 - y + 1 = 0.$$

This will be our elementary object, with $f(x, y) = x^2 - y + 1$. Let us also consider an Euclidean motion g_t parametrized by t

$$g_t = \begin{pmatrix} \frac{-2t+t^2}{2-2t+t^2} & \frac{-2+2t}{2-2t+t^2} & -t^2 \\ \frac{2-2t}{2-2t+t^2} & \frac{-2t+t^2}{2-2t+t^2} & t-t^3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying this motion to our parabola will yield a new parabola for each t . The implicit equations of our system can be computed as $f_t = \phi(g_t) = f \circ g_t^{-1}$. We obtain

$$\begin{aligned} f_t = & \frac{1}{(t^2 - 2t + 2)^2} (t^8 - t^7 - 2t^6x + 4t^5x + 4t^5y - 3t^5 + t^4x^2 + 4t^4x \\ & - 5t^4y + 11t^4 - 4t^3x^2 - 4t^3xy - 14t^3x - 4t^3y - 14t^3 + 4t^2x^2 \\ & + 12t^2xy + 14t^2x + 4t^2y^2 + 10t^2y + 12t^2 - 8txy - 8tx - 8ty^2 \\ & - 4ty - 8t + 4x + 4y^2 + 4). \end{aligned}$$

The resulting motion can be seen in figure 3.1.

Now we want to compute the envelope of our system $\mathcal{F} = \{[x, y] \in \mathbb{R}^2 \mid f_t(x, y) = 0\}$. As seen from the main theorem, we do not actually need the equations f_t , we only need f and the mysterious equation $d_{\text{id}}\phi(\gamma_t)$ to compute the preimage sets. As discussed above, $d_{\text{id}}\phi$ is a linear map from $\mathfrak{g}$ to Q . As discussed, we first need to compute the images of generators of $\mathfrak{g}$ and then express γ_t with respect to said generators.

In our case the Lie algebra $\mathfrak{g}$ is $\mathfrak{se}(2)$. Its dimension is 3 just like its corresponding Lie group, and it is generated by

$$\gamma_r = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \gamma_{tx} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \gamma_{ty} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

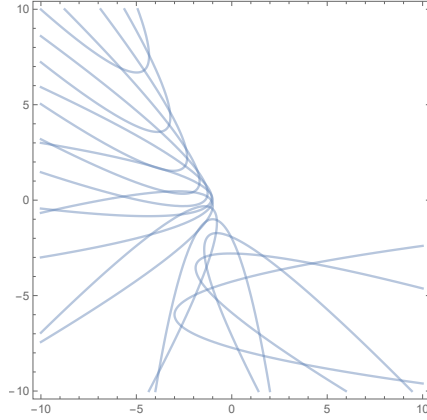


Fig. 1: Motion of parabola for $t \in (-2, 2)$.

The first basis element is the derivative of rotation, the other two correspond to translations in $\mathbb{R}^2$, see [2] for details.

To compute the images of these elements, we need to obtain and then differentiate the equation of parabola undergoing the specific motion. In the case of rotation, we pick

$$r_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

compute the corresponding parabola as $\phi(r_t)$ and obtain

$$(x \cos(t) + y \sin(t))^2 + x \sin(t) - y \cos(t) + 1.$$

Since the unit element of $\mathbb{SE}(2)$ corresponds to $t = 0$, we differentiate the function at 0 to obtain

$$d_{\text{id}}\phi(\gamma_r) = x(1 + y).$$

Analogical computation for translations yields

$$\begin{aligned} d_{\text{id}}\phi(\gamma_{tx}) &= -2x, \\ d_{\text{id}}\phi(\gamma_{ty}) &= 1. \end{aligned}$$

Now, we are only left to compute γ_t and its coefficients in the basis. We obtain

$$\gamma_t = \frac{2}{t^2 - 2t + 2}\gamma_r + \frac{4t^3 - 2t^2 - 2t + 2}{t^2 - 2t + 2}\gamma_{tx} + \frac{-3t^4 + 6t^3 - 3t^2 + 2t}{t^2 - 2t + 2}\gamma_{ty}.$$

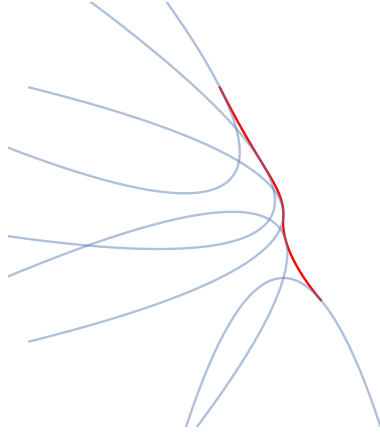


Fig. 2: The envelope of system of parabolas with $t \in (-1, 1)$.

Thus, the resulting equation is

$$\frac{2}{t^2 - 2t + 2}x(1+y) + \frac{4t^3 - 2t^2 - 2t + 2}{t^2 - 2t + 2}(-2x) + \frac{-3t^4 + 6t^3 - 3t^2 + 2t}{t^2 - 2t + 2} = 0.$$

Solving this alongside $f(x, y) = x^2 - y + 1 = 0$ yields the preimage set $\bar{\chi}_t$. Applying the transformation g_t to the set then yields the characteristic set and subsequently the envelope χ , which will be a parametric curve, parametrized by t . Its expression is too long to be included here, but you can see it in figure 3.1.

The most important part of this computation is its great generality. If we change the motion of our parabola, the only thing we have to recompute are the coefficients of γ_t (as long as our motion lies in $\mathbb{SE}(2)$). For $\gamma_t = k_r(t)\gamma_r + k_{tx}(t)\gamma_{tx} + k_{ty}(t)\gamma_{ty}$ we obtain the equation

$$k_r(t)x(1+y) - 2k_{tx}(t)x + k_{ty}(t) = 0.$$

Expressing $y = x^2 + 1$ from the definition of our elementary parabola, we are in fact solving the cubic equation

$$2k_r(t)x^3 + (3k_r(t) - 2k_{tx}(t))x + k_{ty}(t) = 0$$

with parameter t . The resulting preimage set can then be described as a curve

$$\chi_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix},$$

where x_t is the solution to the equation above and $y_t = x_t^2 + 1$.

3.2 Moving plane in space

Let us consider a plane in space defined as $\{[x, y, z] \in \mathbb{R}^3 \mid z = 0\}$, i.e. $f(x, y, z) = z$. Once again we will consider the Euclidean motion, this time in space. The $g_t \subseteq \mathbb{SE}(3)$ will be given by rotation about vector $(t + 2, 3t^2 - 3t + 2, 4t^2 + t + 1)$ by angle $\arccos((t^2 - 1)/(t^2 + 1))$ and translation by vector $(-t^2, 1 + 3t, -2t^3)^T$. The resulting 4×4 matrix is too big to be included here.

Now similarly to precious case, we want to compute the preimage sets. This time the Lie group $\mathbb{SE}(3)$ has dimension 6, so we obtain six basis elements of $\mathfrak{se}(3)$ – three correspond to rotations about axes x, y, z and three to translations in direction of these axes. The resulting basis is

$$\begin{aligned} \gamma_{rx} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \gamma_{ry} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \gamma_{rz} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \gamma_{tx} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \gamma_{ty} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \gamma_{tz} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The images of basis elements under $d_{\text{id}}\phi$ are

$$\begin{aligned} d_{\text{id}}\phi(\gamma_{rx}) &= -y, & d_{\text{id}}\phi(\gamma_{ry}) &= -x, & d_{\text{id}}\phi(\gamma_{rz}) &= 0, \\ d_{\text{id}}\phi(\gamma_{tx}) &= 0, & d_{\text{id}}\phi(\gamma_{ty}) &= 0, & d_{\text{id}}\phi(\gamma_{tz}) &= -1. \end{aligned}$$

Since we are mapping a linear space of dimension 6, we would expect the image to also have dimension 6. However, this is not the case here. The zeros which we computed above correspond to the symmetries of our elementary plane $z = 0$. Neither the rotation about z axis, nor the translations in the direction of x and y axes, change the position of our plane. Thus they do not affect the resulting envelope.

In general case we will get some coefficients a, b, c (depending on t), that will yield the second equation of the form

$$-ax - by - c = 0,$$

which is the implicit equation of an arbitrary plane perpendicular to our elementary plane. Thus the preimage curves will be lines in said plane.

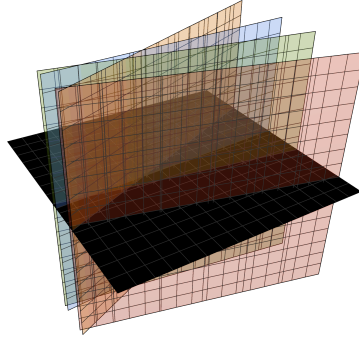


Fig. 3: The elementary plane (black) along with several planes produced by equation $d_{\text{id}}\phi(\gamma_t) = 0$.

In the case of our motion g_t we obtain the equation

$$\frac{1}{(t^2 + 1)(5t^2 - t + 3)^2}(-12 + 50t - 48t^2 + 160t^3 + 38t^4 + 48t^5 + 288t^6 - 60t^7 + t^8 + (-10 + 40t - 40t^2 + 60t^3 - 30t^4)X + (22 + 12t + 28t^2 + 20t^3)Y) = 0.$$

Adding the equation $z = 0$ allows us to compute the preimage curve. The parametrization of resulting line is straightforward, but too long to be included here. The plane $z = 0$ along with the perpendicular planes used for computation of characteristic lines can be seen in figure 3.2.

Applying g_t to the preimage curves will yield the resulting envelope. Since it is a union of lines undergoing Euclidean motion, the envelope is a ruled surface. It can be seen in figure 3.2.

4 Conclusion

We connected the theory of Lie groups to the description of envelope surfaces. Using the theory developed by [5], we were able to define a suitable computation method in section 2, that exploits the inherent symmetry and linearity in computation of characteristic sets.

In section 3 we applied this method to new examples. While the computational concept is not brand new (see [5]), we were able to extend its use beyond surfaces in $\mathbb{R}^3$.

In the future, we would like to apply the method to objects in other

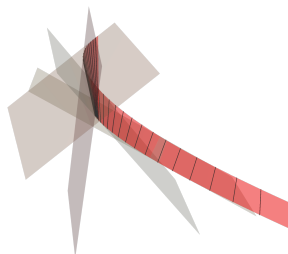


Fig. 4: Several instances of the moving plane (gray) along with their envelope (red).

dimensions and extend the research into projective setup. This would be the first steps in generalizing the method for an arbitrary setting.

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Limits of GeoGebra by solving a problem

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Abstract. In the text we demonstrate a geometry problem that is closely related to the well-known Wallace-Simson theorem. This problem turns out to be quite difficult for some software tools incorporated in the DGE software. The computation of a searched locus by the software works well in our case, but the problem must be formulated in a specific way.

In addition to the conclusions attained by DGE software, we present computer-aided analytical solutions in the second half of paper. It is shown that the use of non-degeneracy conditions before elimination leads to a significant reduction of the computation time.

Keywords: GeoGebra, elimination, locus equation, Wallace-Simson theorem.

1 Introduction

The paper serves as a demonstration of what can be discovered about a problem through human-computer collaboration, thus easing the path to a synthetic solution. It is a concrete example of the topic discussed in the paper. The second goal was to show not only the capabilities of GeoGebra tools, but also its current limitations. In this paper we present a way how to construct a locus of point if a Boolean condition has to be satisfied, which expresses a fact that two lines form an angle of a given magnitude (we mean here the magnitude of a constructible angle).

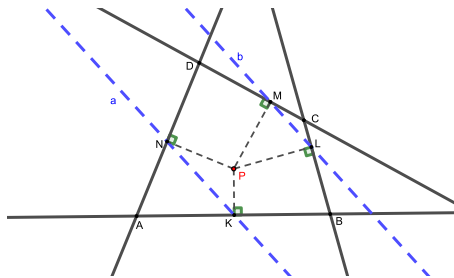


Fig. 1: Determine the locus of P such that $KN \parallel LM$.

As for the computer algebra software CoCoA [3], besides the general

demonstration of the solution method, two points are emphasized. The first has to do with the fact that currently a locus equation cannot be factorized in CAS software without a human activity. The second point shows the fact that when solving a problem using CAS, prescribing non-degenerate conditions before eliminating variables has a significant impact on the computation time, see also [1, 2].

2 Special cases of a problem

A special case of a problem is as follows:

Let $ABCD$ be a quadrilateral and K, L, M, N feet of perpendiculars from a point P to lines AB, BC, CD, DA . Determine the locus of the point P such that KN is parallel to ML , see Fig. 1.

Using GeoGebra command `LocusEquation(AreParallel(KN,LM),P)` we get a circle c_1 passing through the points A, C , see Fig.2.

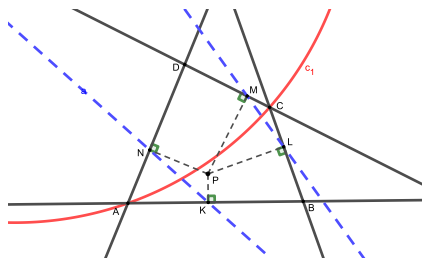


Fig. 2: The locus of P is a circle c_1 through the points A, C .

Construction of the circle c_1 :

The circle c_1 must pass through the so called *Miquel* point Q which is the

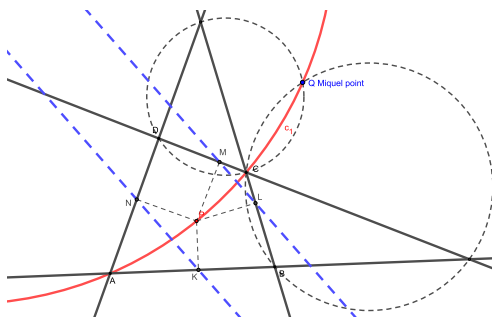


Fig. 3: The circle c_1 is passing through A, C and the Miquel point Q .

point, such that feet of perpendiculars from Q to the sides of a quadrilateral $ABCD$ are collinear [4]. Now the classical construction of the locus follows. The circle c_1 is determined by three points A, C and the Miquel point Q , see Fig. 3.

Another special case.

Let $ABCD$ be a quadrilateral and K, L, M, N feet of perpendiculars from a point P to lines AB, BC, CD, DA . Determine the locus of the point P such that KN is perpendicular to ML , see Fig. 4.

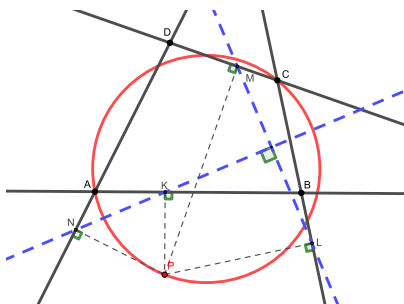


Fig. 4: Determine the locus of the point P such that KN is perpendicular to ML .

Using GeoGebra command `LocusEquation(ArePerpendicular(KN, LM), P)` we surprisingly get again a circle c_2 through the points A, C including its equation, see Fig. 5.

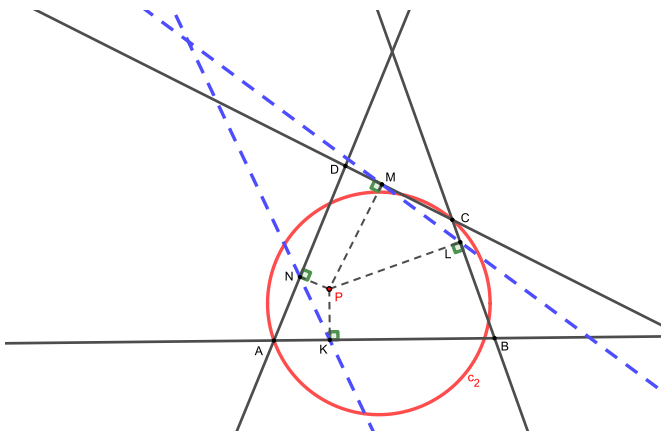


Fig. 5: The locus of the point P is a circle c_2 .

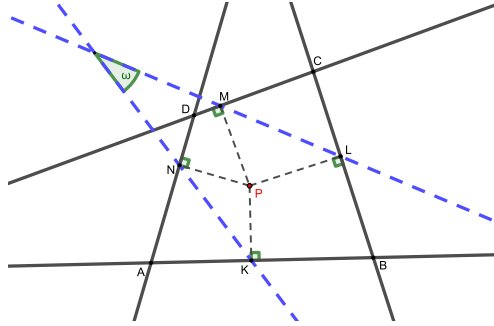


Fig. 8: Determine the locus of P such that KN makes with ML a given angle ω .

Using GeoGebra command for $\omega = 30^\circ$

`LocusEquation(Angle(KN,LM)==30°,P)`

gives no output.

Similarly to the previous case, we construct circle c_3 such that $\varphi = \gamma + 30^\circ$ is perimeter angle above segment AC . The construction works as well!

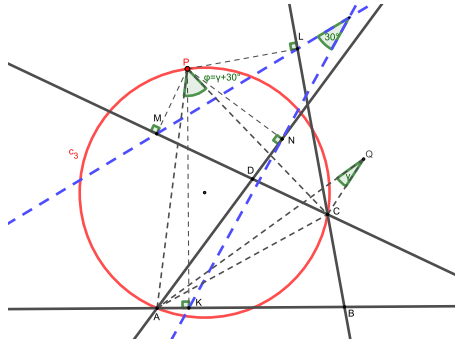


Fig. 9: $\varphi = \gamma + 30^\circ$.

Classical construction of the locus circle c if KN makes with LM the angle ω , where

$$\angle APC = \varphi = \gamma + \omega \quad (1)$$

is similar to the previous case. It turns out that the relation (1) is correct, we proved it classically as well.

with

$$M = (p^2 + q^2)^2(E^2 - H) + 2(p^3 + pq^2)(uH - EF) + 2(p^2q + q^3)(vH - EG) + p^2(F^2 - (u^2 + v^2)H) + 2pqFG + q^2(G^2 - (u^2 + v^2)H),$$

where

$$\begin{aligned} E &= auw - u^2w + v^2w - aw^2 + uw^2 + avz - 2uvz - az^2 + uz^2, \\ F &= au^2w - u^3w + av^2w - uv^2w - auw^2 + u^2w^2 + v^2w^2 - u^2vz - v^3z - auz^2 + u^2z^2 + v^2z^2, \\ G &= u^2vw + v^3w - avw^2 + au^2z - u^3z + av^2z - uv^2z - avz^2, \\ H &= c^2(w^2 + z^2)((u - w)^2 + (v - z)^2)((u - a)^2 + v^2). \end{aligned}$$

We can assume that $z(av - vw - az + uz) \neq 0$ otherwise the quadrilateral $ABCD$ degenerates. Now the locus equation of the point P may be written in the form

$$M = 0.$$

To make the elimination process faster, we can apply this non-degeneracy condition

$$z(av - vw - az + uz)t - 1 = 0,$$

where t is a slack variable, *before* starting the calculation [5].

In 1 hour and 30 minutes we obtain the locus equation directly in the form $M = 0$. Now the resulting polynomial M has 901 terms. $M = 0$ is a fourth-degree equation, but further factorization using a computer is not possible.

The locus equation $M = 0$ may be written using CAS and human intervention in the form of a product

$$(p^2 - 2ps_1 + q^2 - 2qt_1) \cdot (p^2 - 2ps_2 + q^2 - 2qt_2) = 0,$$

where

$$\begin{aligned} s_1 &= (c(EF - uH) + vsH)/(2c(E^2 - H)), \\ t_1 &= (c(EG - vH) - usH)/(2c(E^2 - H)), \\ s_2 &= (c(EF - uH) - vsH)/(2c(E^2 - H)), \\ t_2 &= (c(EG - vH) + usH)/(2c(E^2 - H)) \end{aligned}$$

with

$$s = \sin \omega, c = \cos \omega \text{ and } c \neq 0, E^2 - H \neq 0.$$

We see that the locus M consists of two circles centered at $[s_1, t_1]$ and $[s_2, t_2]$ passing through the points A and C . Note that along with the prescribed angle ω corresponding to $\cos \omega$, the solution also includes the angle $\pi - \omega$, which corresponds to $-\cos \omega$. However, for both angles, the lines KN and LM form the angle $\min(\omega, \pi - \omega)$. Thus, the solution

contains two circles.

Example: For $\omega = 60^\circ$ and $a = 4, u = 3, v = 3, w = 1$ and $z = 2$ we have

$$p^4 + 2p^2q^2 + q^4 + 6p^3 - 18p^2q + 6pq^2 - 18q^3 - 66p^2 + 96pq + 6q^2 = 0.$$

Factorization in Maple by **AFactor** (over constructible reals) yields

$$(p^2 + q^2 + (3 + 5\sqrt{3})p - (9 + 5\sqrt{3})q)(p^2 + q^2 + (3 - 5\sqrt{3})p - (9 - 5\sqrt{3})q) = 0, \quad (2)$$

see Fig. 11. We see that (2) represents two circles.

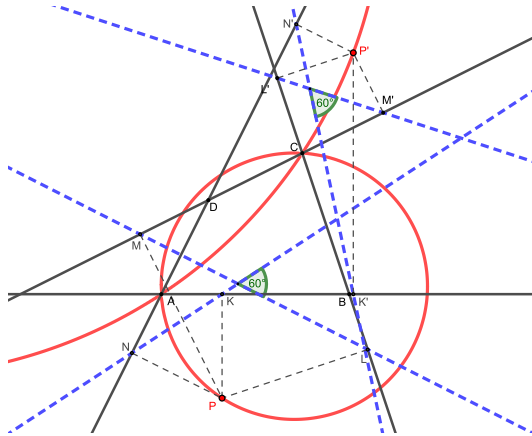


Fig. 11: Locus of P consists of two circles.

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3D pen-supported geometry learning in secondary schools

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Abstract. The study of properties of three-dimensional objects and their mutual relationships often proceeds via planar projections, that is, two-dimensional representations. For some students, however, such 2D projections do not evoke an adequate mental model of the corresponding 3D object. The growing accessibility of 3D printing technologies introduces new opportunities for teaching three-dimensional geometry. This paper reports research findings demonstrating substantial benefits of employing 3D pens in the teaching of solid geometry. It also presents new instructional materials for classroom work with a 3D pen.

Keywords: 3D pen, solid geometry, teaching, education.

1 Introduction

Psychological research indicates that tactile-kinesthetic learning methods are highly effective for students. Learning is particularly effective when it includes observation, measurement, and hands-on manipulation of physical objects, an approach known as embodied learning (Hall and Nemirovsky [3], Shapiro [11]). Nevertheless, current classroom practice often provides too few opportunities for such activities. The 3D pen not only allows students to physically construct models of basic geometric solids but also enables them to draw directly in space – lines, planes, and cross-sections of solids – capabilities that were previously unavailable. Published findings suggest that using such tools improves students' conceptual understanding and retention in spatial (solid) geometry.

2 Using 3D pen in Solid Geometry Education

3D printing technology is increasingly used across diverse educational domains. In physics, students can create models of various mechanical devices. In chemistry, 3D printing can be employed to produce molecular models of chemical compounds. Medical students can also design models of organs and cells. In geography classes, students can construct topographic maps and urban planning models. In engineering and technology, students can prototype designs for technical apparatus, thereby enhancing their capacity for visualization and experimentation. Volume [2] presents a synthesis of findings from the International Symposium on 3D Printing in Mathematics Education.

2.1 The 3D Pen as an Educational Tool

A 3D pen is an innovative technology that enables the creation of spatial objects from thermoplastic materials. The pen extrudes heated plastic that cools almost immediately into a solid, stable structure, thereby allowing the freehand fabrication of three-dimensional objects. It uses a plastic filament made of polylactic acid (PLA). Compared with a 3D printer, the 3D pen is more affordable and requires no software to operate. It thus makes it possible to produce 3D models in home or school settings with minimal setup. In the instructional context, it is therefore natural to employ it for teaching solid geometry.

2.2 Related Work

Several researchers have explored the use of 3D pens in geometry instruction. Ng and Ye [8] present the first author's five-year research program on the use of 3D pens in mathematics classrooms. In the first of these studies, Ng and Sinclair [6] describe using a 3D pen to teach properties of graphs of functions. Students learned concepts such as the tangent line and slope through physical manipulation of the graph of a function. They then used a 3D pen to draw a curve together with the coordinate axes and rotated the x -axis, thereby obtaining a solid of revolution and improving their visualization of how it is formed. The study by Ng and Ferrara [4] describes a lesson design for teaching and learning the target properties of prisms, pyramids, and cross-sections of 3D solids using 3D pens alongside glass solids filled with water, where the water level represented a cutting plane. The paper by Ng, Shi, and Ting [5] compares differences in geometry learning outcomes in two technology-enhanced environments. Two groups shared an almost identical lesson procedure except for the technology used in class. The first group used 3D pens to explore the properties of prisms and pyramids, while the second group used a pre-made dynamic geometry environment (DGE). This DGE displayed a 3D solid on the computer screen, allowing students to perform basic transformations such as rotation and translation.

The findings of this five-year research program can be summarized as follows: the studies highlight the unity of mind-body and body-tool interactions in the act of making something (i.e., mathematics learning as embodied making). 3D pens, as a form of maker technology – together with embodied making – supported students' investigation, visualization, and learning of geometric relationships in significant ways. Students using 3D pens demonstrated better retention of the properties of 3D solids.

2.3 Solid Geometry with 3D Pen

All of the aforementioned studies primarily focused on younger students (aged 11–12). In contrast, this research also focuses on older students, who

often experience difficulties when solving more complex problems in solid geometry. The study of the properties and mutual relationships of three-dimensional objects often proceeds via planar projections. It is difficult for some students to realize that not all lines that intersect on paper actually have a point of intersection in the original three-dimensional space. The author of this paper supervised several completed theses aimed at developing instructional materials for teaching solid geometry using 3D pens in secondary schools.

In Redechová [10], we focused on planar cross-sections of a cube, typically taught in the seventh year (septima, age 17) of secondary school. Our objective was to demonstrate the range of polygons that can arise as the intersection of a cube with a sectioning plane (fig. 1). We deliberately selected problem statements from textbooks currently used in Slovakia, aiming to make the teaching method attractive to teachers rather than burdensome. For these examples, we developed templates for constructing 3D models of cube–plane intersections (fig. 2). Using these models, students were able to verify the correctness of their solutions and examine the results from different viewpoints. The ability to perform physical rotations proved valuable, since mental rotation is often challenging for students (Bruce and Hawes [1]).



Fig. 1: A set of 3D pen models illustrating the variety of polygons that can occur as cross-sections of a cube. Source: Redechová [10], p. 70.

The thesis by Piačková [9] aims to support students' spatial visualization in analytic geometry problems involving the computation of line-segment lengths and angles between planes and lines. This topic is typically taught in the seventh class (septima, age 17) of secondary school. The proposed tasks focus on using the dot product, the Pythagorean theorem, and the law of cosines. The templates were designed to enable the

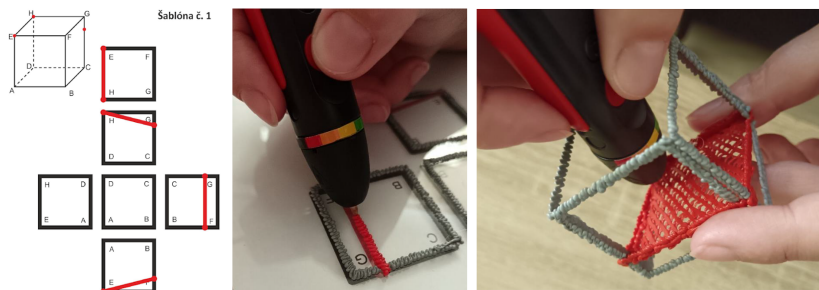


Fig. 2: Step-by-step construction of a cube–plane intersection model using a 3D pen: from the printed template to the completed 3D structure. Source: Redechová [10], pp. 67, 69, 70.

constructed models to clearly illustrate the spatial relationships among the figures whose properties students are expected to determine. Each template set includes a paper ruler, allowing students to verify the lengths of calculated line segments (fig. 3). Because paper is flexible, it is possible to perform measurements even within the interior of the cube. The measurements are not entirely precise, as the quality of the constructed model depends on the students' manual skills. Nevertheless, they are sufficient to reveal substantial deviations and to assist students in developing an awareness of the spatial position of the measured line segment. Students can situate the entire model within a coordinate system.

Trubačová [12] focused on younger students attending the second year (sekunda, age 13) of secondary school, who were learning their first projection method – oblique parallel projection. Students learn to draw basic 3D solids, such as cubes, rectangular prisms, and pyramids, in cabinet projection. Therefore, we created a template containing squares, rectangles, and triangles that can be used to build these basic solids. Thanks to the hatching on the cube's faces, the hidden edges are shown clearly as dashed lines, which is not achievable with traditional paper models students usually make at school.

Physical models enable students to explore the fundamental properties of geometric shapes through tactile and visual experience. By manipulating the models, students can literally feel what it means for edges or planes to be parallel or perpendicular, transforming abstract spatial relationships into concrete sensory experiences. This multimodal perception greatly supports the development of accurate mental representations of solids. The perception of a cube's face depends on the viewing angle (by sight one perceives a parallelogram; by touch, a square). Building these

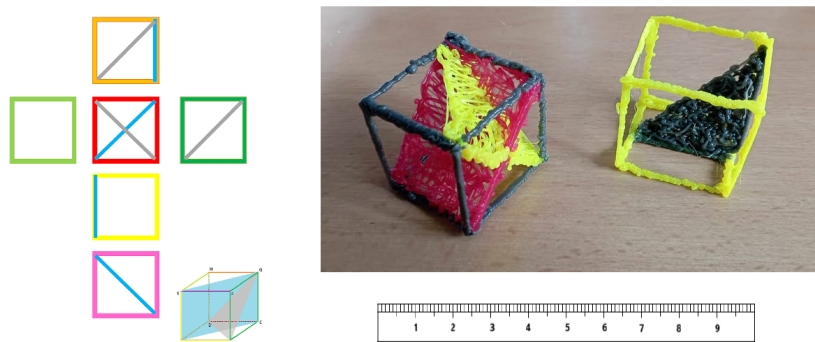


Fig. 3: Classroom models created with a 3D pen: the left model, together with its corresponding template, illustrates the perpendicularity of two planes, while the right model, together with a paper ruler, serves to verify calculated lengths and areas using the Pythagorean theorem. Source: Piačková [9], pp. 58, 71, 78.

mental connections is very important at this age (fig. 4).

The thesis also includes proposed communication scenarios that represent model dialogues between the teacher and the student, designed as a methodological framework for conducting mathematics lessons that employ 3D pen technology. These scenarios are structured to foster guided discovery and student reflection through question sequences that encourage independent formulation of geometric concepts, relationships, and projection rules. They include progressively graded interactions – from motivational and situational prompts to analytical and verification-oriented questions that reinforce and apply newly acquired knowledge during task completion. The dialog structure simultaneously strengthens elements of the constructivist approach, promotes students' reasoning and argumentation skills, and creates space for mutual discussion and cooperative learning.

2.4 Application in the Classroom

All three authors of [10], [9], and [12] were teaching mathematics while studying, which enabled them to test the designed teaching aids directly in classrooms.

In Redechová [10], two parallel classes were involved in the experiment. Each class was assigned general tasks for spatial imagination. After analyzing the results of these tasks, the class that achieved weaker results was identified. This class (11 students) was selected as the experimental group in which teaching using a 3D pen was implemented. In the second class,

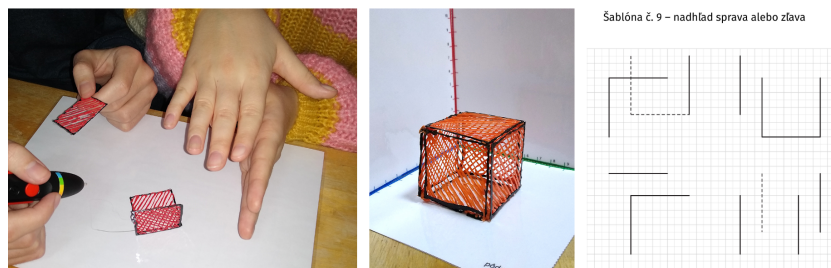


Fig. 4: Classroom activity on cabinet projection: on the left, students construct a cube model and use hand gestures to indicate the perpendicularity of its faces; in the center, the completed cube is placed in a coordinate system – hatched faces make it possible to distinguish visible and hidden edges; on the right, the worksheet template on which students supplemented the diagram with visible (solid) and hidden (dashed) edges to produce the cube's cabinet projection (left or right view), using the constructed model for reference. Source: Trubačová [12], pp. 36, 44, 64.

designated as the control group (12 students), teaching was conducted traditionally, using paper-based drawing. After the end of the teaching, students in both groups were given further tasks to develop their spatial imagination, including a written test. Although the sample of students was too small for a statistically relevant assessment of the effectiveness of teaching with 3D pens compared to standard methods, the experimental group demonstrated greater attention, anticipation, and active engagement during lessons.

In Piačková [9], 18 students worked together in groups of three and took turns using the 3D pen to construct geometric models. According to their responses in the post-lesson questionnaire, they particularly appreciated the opportunity to verify their calculated results by directly comparing them with the physical model. The tangible representation made it easier for them to identify geometric properties, such as which of the constructed triangles were right-angled, although they subsequently had to provide a formal mathematical justification. The teacher's observations also revealed that even students who are usually passive during math lessons were actively involved in this collaborative environment. Working in groups and using the 3D pen stimulated motivation, encouraged students to participate in problem solving, explore alternative approaches, and engage in independent learning – often without realizing it. Based on the observations, the new approach was perceived as more attractive and innovative compared with traditional teaching methods and was a good

complement to paper-based constructions.

The experimental teaching in Trubačová [12] was implemented in two student groups comprising 12 and 15 seventh-grade learners (equivalent to the second year of secondary school). Throughout the sessions, a high level of engagement was observed, accompanied by spontaneous nonverbal expression of understanding. Students unconsciously used hand gestures to represent geometric concepts – for example, illustrating a cube's face with the palm of the hand, indicating parallel and perpendicular lines with appropriately positioned fingers, or marking points with fingertips. These gestures served as clear indicators of internalized comprehension of spatial relationships. The use of two-color filaments further enhanced visual clarity, enabling students to distinguish geometric elements such as edges and faces more easily. Many students expressed enthusiasm for working with the 3D pen, noting that they preferred creating geometric figures in this way rather than with a pencil, as it allowed them to physically touch and examine their 'drawings' from multiple perspectives. Even those who struggled to complete a full model – whether due to breakage or other technical issues – remained engaged by borrowing models from classmates and continuing to solve related tasks collaboratively. Students were evidently more focused and motivated when working with objects they had created themselves than when merely viewing teacher-prepared models.

3 Conclusion

Across all conducted studies, the use of 3D pen technology in geometry instruction has demonstrated strong potential to enhance students' spatial reasoning, conceptual understanding, and engagement. The combination of tactile manipulation and visual observation enables learners to connect abstract geometric principles with concrete experience, thereby reducing misconceptions and improving the accuracy of spatial visualization. Collaborative group work with 3D pens promotes active participation, self-directed learning, and peer-to-peer explanation, even among typically passive students. Teachers have also emphasized the accessibility and intuitive nature of the tool, which requires minimal technical preparation while offering high didactic value. The results collectively suggest that integrating 3D pen activities into mathematics education can effectively bridge the gap between theory and practice. Future research should focus on long-term impacts and evaluation of the effectiveness of using the proposed teaching aids across multiple schools. We plan to develop additional teaching materials for other areas of solid geometry and to create a platform where these materials will be freely available for download, together with usage instructions. All of these measures are aimed at making the study of geometry and methods of descriptive geometry more attractive.

Acknowledgements

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Aplikácia rotačných plôch v architektúre

Surfaces of revolution – application in architecture

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Abstract. Surfaces of revolution are frequently used in architectural practice. This paper presents them as the theme for a team-based graphical project assigned to students at the Faculty of Architecture and Design (STU Bratislava). The project's goal was to design a structure composed of multiple types of surfaces of revolution sharing a common axis. Students were tasked with graphically developing the design using projection methods based on orthogonal projection, and then creating a 3D visualization in a virtual environment with appropriate modeling software. As an optional bonus, students could also complement their project with a 3D printed model.

Keywords: surface of revolution, Monge's method, axonometry, 3D digital model, 3D printed model

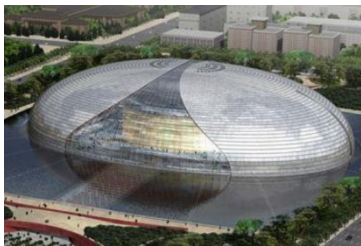
Kľúčové slová: rotačná plocha, Mongeova projekcia, axonometria, digitálny 3D model, tlačný 3D model

1 Využitie rotačných plôch v architektúre

Rotačné plochy majú využitie v architektúre pre svoje rozmanité praktické aj estetické vlastnosti. Často ich vidíme napr. ako zastrešenia rozličných typov budov (chrámy, planetária, štadióny) vo forme kupoly (polguľová, parabolická alebo eliptická). Tvar valca môžeme vidieť na stĺpoch, vežiach, rotundách či celých budovách tohto tvaru. Ich hladký a oblý tvar je vysoko aerodynamický, čím lepšie odoláva vetru. Budovy s tvarom kupoly sú tiež energeticky úspornejšie. Oblúbeným a efektívnym v modernej architektúre je tiež elegantný tvar jednodielneho rotačného hyperboloidu (chladniarske veže, rozhľadne, nosné prvky), ktorý je možné postaviť len pomocou úplne rovných tyčí (keďže je to zároveň priamková plocha) (Obr.1).

2 Rotačné plochy vo výučbe

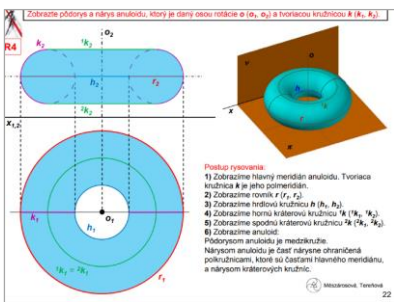
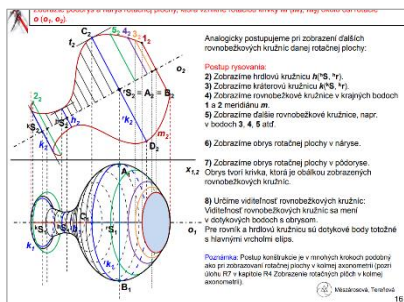
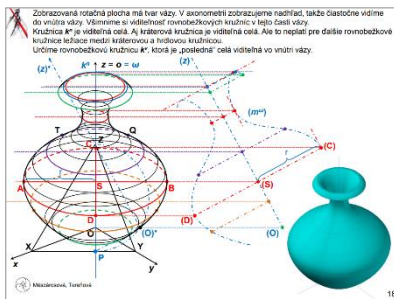
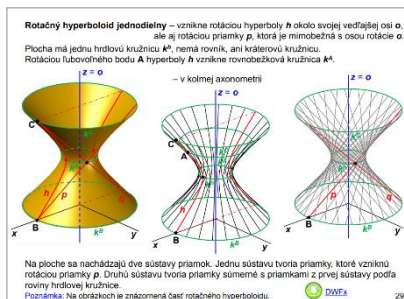
Študenti FAD a SvF STU sa s rotačnými plochami oboznámia v rámci predmetu Deskriptívna geometria v časti Plochy technickej praxe. Naučia sa základné



Obr. 1: vľavo - Paul Andreu [3], Národné divadlo, Peking, Čína, vpravo - Oscar Niemeyer [2], Katedrála v Brazílii

pojmy o rotačnom pohybe a princípe vzniku rotačných plôch. V rámci ich klasifikácie sa zoznámia s rôznymi typmi rotačných plôch ako sú: priamkové, cyklické, kvadratické a všeobecné. Aplikujú tiež svoje znalosti zobrazovacích metód a naučia sa zostrojiť ich obraz v kolmom rovnobežnom premietaní (Mongeova projekcia, kolmá axonometria).

Veľmi efektívnou pomôckou pri výučbe a štúdiu sú materiály dostupné ako online učebnica, ktoré majú formu PowerPoint prezentácií [1], (Obr.2).



Obr.2: Ukážky snímok zo študijných materiálov [1]

Doplnkom prezentácii sú počítačové modely (*.dwfx), ktoré umožňujú prezerat' si konkrétny model plochy pri otáčaní v reálnom čase. Okrem teoretických poznatkov tu študenti nájdu aj množstvo príkladov, kde v architektúre a dizajne boli dané plochy aplikované, a ktorými sa môžu inšpirovať.

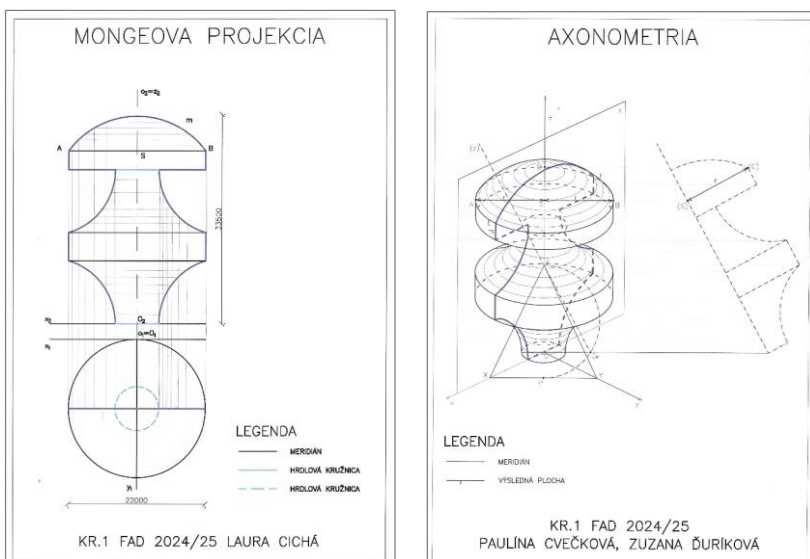
3 Rotačné plochy – téma grafickej práce

Súčasťou celkového hodnotenia študenta v predmete je priebežné hodnotenie, ktoré pozostáva z dvoch teoretických testov a vypracovaní dvoch grafických prác. Ako experiment sme sa rozhodli grafickú prácu na tému Aplikácia rotačných plôch v architektúre zadať ako skupinový projekt.

Cieľom práce bolo vytvoriť návrh stavby alebo jej časti, umiestnenej do zvolenej lokality alebo terénu. Ako hlavný konštrukčný a estetický prvok mohla byť využitá rotačná plocha (celá, alebo len jej časť, alebo len jednotlivé polohy rotujúceho meridiánu).

Navrhnutý objekt mal byť kombináciou aspoň troch rôznych typov rotačných plôch, ktoré majú spoločnú os rotácie, napr. všeobecná rotačná plocha, rotačný valec, rotačný kužeľ, guľa, jednodielny rotačný hyperboloid, rotačný elipsoid, anuloid či iné.

Zadanie malo spĺňať konkrétne požiadavky na grafické spracovanie, ktoré študenti mohli vykresliť ručne a tiež s využitím grafických softvérov na štyri výkresy formátu A4, ktoré následne prepoja:

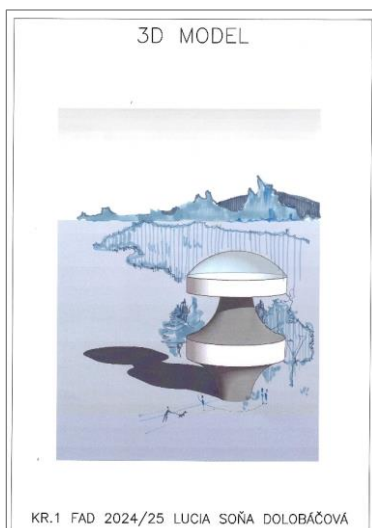


Obr. 3: Ukážka z grafickej práce študentov FAD STU

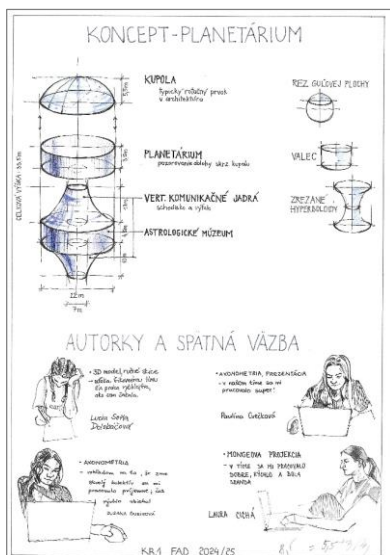
- 1) Zobrazenie objektu v pohľadoch Mongeovej projekcie (pôdorys, nárys), kde vo zvolenej mierke zohľadnia reálne rozmery objektu, ktoré okóujú. Vyznačia tiež obrysové tvoriace kružnice a aj špeciálne rovnobežkové kružnice ak ich plocha obsahuje, taktiež hlavný meridián, os rotácie a súradnicové osi (Obr.3 vľavo, Obr.6 vpravo).
- 2) Zobrazenie objektu v kolmej axonometrii (daný ΔXYZ) pomocou meridiánu zhodného s meridiánom z Mongeovej projekcie. Mali zohľadniť viditeľnosť rovnobežkových kružníc a zobraziť meridiánový rez ležiaci v jednej zo súradnicových rovín. Hotovú axonometriu mali doplniť pauzovacím papierom, na ktorom bude zvýraznený a esteticky v okolí dotvorený výsledný objekt návrhu (Obr.3 vpravo, Obr.4 vľavo, Obr.6 vľavo).
- 3) Spracovanie objektu ako digitálneho 3D modelu vo vhodnom softvérovom programe (napr. AutoCAD, Sketchup, Blender a iné.) s využitím jeho rozmerov a geometrických vlastností, ktorý vo výsledku môže byť zasadený do virtuálneho okolia (Obr.4 vpravo).
- 4) Záverečnou požiadavkou bolo stručne popísať návrh objektu, jeho rozmery, účel, materiály a pod. Z geometrického hľadiska uviesť z akých tipov rotačných plôch je zostavený. Taktiež popísať proces rozdelenia práce v tíme, kde sa vyjadrí každý člen, ako a na čom pracoval (Obr.5 vľavo).

Bola stanovená aj požiadavka nad rámec zadania, za ktorú bolo možné získať bonusové body, a to vytvorenie fyzického modelu objektu, ideálne 3D tlačeneho modelu (Obr.5 vľavo, Obr.7), ktorý tiež mohol byť súčasťou celkového modelu s okolitou krajinou.

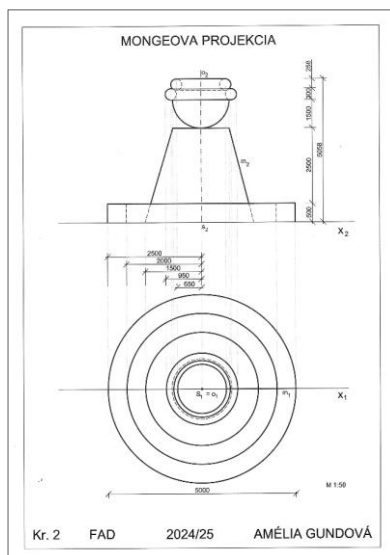
Takto špecifikované zadanie bolo pridelené do 3-4 členných skupín študentov jednotlivých krúžkov.



Obr. 4: Ukážka z grafickej práce študentov FAD STU



Obr. 5: Ukážka z grafickej práce študentov doplnená o 3D tlačný model [4]



Obr. 6: Ukážka z grafickej práce študentov FAD STU

4 Záver

Takto postavené projektové zadanie sa ukázalo pre študentov ako veľmi prínosné vo viacerých aspektoch. Posúva ich zručnosti od izolovaných cviční k riešeniu komplexného priestorového problému, čo je vlastne podstatou budúcej práce architekta. Intenzívne pritom trénujú priestorovú predstavivosť a schopnosť analyzovať zložitejšie 3D formy. Zároveň sa učia ako tieto tvary presne, jednoznačne a technicky správne komunikovať pomocou výkresovej dokumentácie obsahujúcej pohľady, rezy či axonometriu. Zároveň si vyskúšajú aj vzájomnú spoluprácu v tíme, pri ktorej si navzájom kontrolujú prácu, môžu diskutovať o rôznych prístupoch k riešeniu a učia sa prezentovať svoje návrhy. Samotní študenti sa vyjadrili o takomto type tímového zadania veľmi pozitívne.



Obr. 7: Ukážka 3D tlačенých modelov ku grafickým prácam študentov

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On orthogonal trajectories of isoptics

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Abstract. For a given plane curve, consider a one-parameter family of curves (called isoptics) each of which being the locus of points from which the given curve is seen at a constant angle. We present a Cauchy problem, the solution of which can be used to construct the parametric form of the orthogonal trajectories to isoptics of the given oval.

Keywords: isoptic curve, support function, orthogonal trajectory.

1 Introduction

For a plane curve C one can consider locus of the points at which the support lines to C intersect under the fixed angle $\pi - \alpha$. The curve C_α , where $\alpha \in (0, \pi)$, is called an α -isoptic, or simply an isoptic [Philippe de La Hire].

As examples of isoptics given by implicit equations let us recall

- isoptics of the circle $x^2 + y^2 = r^2$:

$$C_\alpha: x^2 + y^2 - \frac{r^2}{\cos^2 \frac{\alpha}{2}} = 0, \quad (1)$$

- isoptics of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ derived in [2]:

$$C_\alpha: \cos \alpha = -\frac{a^2 + b^2 - x^2 - y^2}{\sqrt{(x^2 - a^2 + b^2)^2 + 2y^2(x^2 + a^2 - b^2) + y^4}}. \quad (2)$$

Another, very convenient way to describe the isoptics is the parametrization by the support function of the given curve.

Let us recall the definition of the support function $p(t)$ of a given closed, strictly convex curve C , following [8]. We fix the origin O of the coordinate system in the interior of C . Denote by t the angle formed by the first coordinate axis and the halfline starting from O and perpendicular to the support line to C . Define $p(t)$ as the distance from O to the support line of C perpendicular to e^{it} at $z(t) \in C$ (see Fig. 1). We can parametrize the given curve C , by the formula

$$z(t) = p(t)e^{it} + p'(t)ie^{it} \quad \text{for } t \in [0, 2\pi).$$

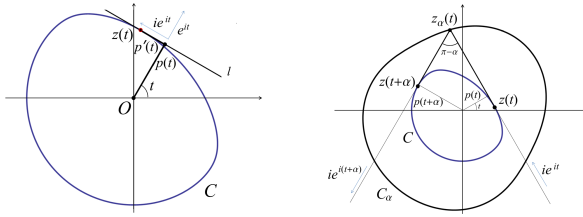


Fig. 1: Support function parametrization for an oval and parametrization of isoptics by the support function of the given curve

For ovals (a simple closed convex plane curves of class C^2 with positive curvature) $p(t) \in C^2$ and the radius of curvature $R(t) = p(t) + p''(t)$ is positive for all $t \in [0, 2\pi)$. The notion of the support function has been recently developed for open curves as well as curves and surfaces represented by polynomial support functions (see [11], [9]).

For C given by the support function parametrization $z(t) = p(t)e^{it} + p'(t)ie^{it}$, an α -isoptic C_α is given by

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t) \cot \alpha + \frac{1}{\sin \alpha} p(t + \alpha) \right) ie^{it}, \quad t \in [0, 2\pi).$$

For more details about this approach see [1]. It was introduced for a given closed, strictly convex curve and studied also for a wider class of curves (see for example [5], [7], [11]).

2 Orthogonal trajectories of isoptics

2.1 Classical approach

Let us recall that the **orthogonal trajectory** is a curve, which intersects any curve of a given pencil of (planar) curves orthogonally.

The classical approach in Cartesian coordinates assumes that we have a pencil of implicitly given curves $F(x, y, c) = 0$ for which we can write a first order ordinary differential equation $F_x(x, y, c) + F_y(x, y, c)y' = 0$ and simplify it by eliminating the parameter c to the form $y' = f(x, y)$. Then we can consider the differential equation $y' = -\frac{1}{f(x, y)}$, which is satisfied by orthogonal trajectories of the given curves.

Examples

1. For isoptics of the circle $C_\alpha: x^2 + y^2 - \frac{r^2}{\cos^2 \frac{\alpha}{2}} = 0$ by straightforward computations we get the following equation for orthogonal trajectories $y' = \frac{y}{x}$ which can be easily solved: $y = cx$, where $c \in \mathbb{R}$.

2. For an ellipse $C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, α -isoptics are given by equation (2).
A differential equation for orthogonal trajectories of those curves is

$$y' = \frac{y}{x} \frac{(a^4 - a^2b^2 + x^2(2b^2 - a^2) + y^2a^2)}{(b^4 - a^2b^2 + x^2b^2 + y^2(2a^2 - b^2))}.$$

2.2 Parametric approach

Now we are going to derive a Cauchy problem, the solution of which can be used to obtain the parametric form of orthogonal trajectories to isoptics. Those results have already been published in [10].

Recalling [1] we can assign the following diffeomorphism to isoptics:

$$F(\alpha, t) = z_\alpha(t) = p(t)e^{it} + \left(-p(t) \cot \alpha + \frac{1}{\sin \alpha} p(t + \alpha) \right) ie^{it}$$

between the rectangle $(0, \pi) \times (0, 2\pi)$ and the exterior of C without one halfline.

Now let us consider orthogonal trajectories to isoptics as curves $\gamma(\alpha) = F(\alpha, t(\alpha))$, where $\alpha \in [0, \pi)$, each of which starts at a point on the oval C and have one common point with each isoptic. There $t(\alpha)$ is a function of variable α and allows us to move along the isoptic C_α .

We are looking for such a curve γ , for which the tangent vector $\gamma'(\alpha)$ is perpendicular to the tangent vector to C_α in the intersection point of those curves. This condition can be written in the form

$$\langle \gamma'(\alpha), z'_\alpha(t(\alpha)) \rangle = 0.$$

To consider the above condition we use standard notation (as seen in Fig. 2) introduced for isoptics in [1] after the following slight extention, which we proposed in [10]. For $(\alpha, t) \in [0, \pi) \times \mathbb{R}$ let us define

$$F(\alpha, t) = \begin{cases} z_\alpha(t), & (\alpha, t) \in (0, \pi) \times \mathbb{R}, \\ z(t), & (\alpha, t) \in \{0\} \times \mathbb{R}. \end{cases} \quad (3)$$

$$\lambda(\alpha, t) = \begin{cases} \frac{(p(t+\alpha) - p(t) \cos \alpha - p'(t) \sin \alpha)}{\sin \alpha}, & (\alpha, t) \in (0, \pi) \times \mathbb{R}, \\ 0, & (\alpha, t) \in \{0\} \times \mathbb{R}, \end{cases} \quad (4)$$

$$\mu(\alpha, t) = \begin{cases} -\frac{(p(t) - p(t+\alpha) \cos \alpha + p'(t+\alpha) \sin \alpha)}{\sin \alpha}, & (\alpha, t) \in (0, \pi) \times \mathbb{R}, \\ 0, & (\alpha, t) \in \{0\} \times \mathbb{R}, \end{cases} \quad (5)$$

$$\nu(\alpha, t) = \begin{cases} \frac{\mu(\alpha, t)}{\sin \alpha}, & (\alpha, t) \in (0, \pi) \times \mathbb{R}, \\ -\frac{1}{2}R(t), & (\alpha, t) \in \{0\} \times \mathbb{R}, \end{cases} \quad (6)$$

$$\rho(\alpha, t) = \begin{cases} \frac{(p(t) \sin \alpha - p'(t) \cos \alpha + p'(t+\alpha))}{\sin \alpha}, & (\alpha, t) \in (0, \pi) \times \mathbb{R}, \\ R(t), & (\alpha, t) \in \{0\} \times \mathbb{R}. \end{cases} \quad (7)$$

has a unique solution.

Theorem 2.3 *Orthogonal trajectories to isoptics of an oval C , where C is parametrized in terms of the support function p of class C^3 , are the curves parameterized by functions $\gamma_{\tau_0}: [0, \pi) \rightarrow \mathbb{R}^2$ for $\tau_0 \in [0, 2\pi)$, defined by*

$$\gamma_{\tau_0}(\alpha) = F(\alpha, t(\alpha)),$$

where $t: [0, \pi) \rightarrow \mathbb{R}$ is the solution to the Cauchy problem

$$\begin{cases} t'(\alpha) = H(\alpha, t(\alpha)), & \alpha \in (0, \pi), \\ t(0) = \tau_0. \end{cases} \quad (11)$$

The proofs of Theorem 2.2 and Theorem 2.3 are available in [10].

Remark It can be noticed that the tangent to the orthogonal trajectory $\gamma_t(\alpha)$ is the line passing through points $P(t)$ and $z_\alpha(t)$ indicated in Fig. 2. The direction vector of this line is $\vec{w} = \rho e^{it} + \lambda i e^{it}$. From (8) and (9) we have $\gamma'_t(\alpha) = A \cdot \vec{w}$, where $A = -\frac{\nu\lambda}{\lambda^2 + \rho^2}$. The authors would like to thank Prof. G. Weiss and Prof. H. Stachel for their advice about this fact.

Examples

1. For the circle $C: x^2 + y^2 = r^2$ the support function is $p(t) = r$ and the parametrization of isoptics is $z_\alpha(t) = r e^{it}$. We can easily compute the function $H(\alpha, t) = -\frac{1}{2}$ for $(\alpha, t) \in [0, \pi) \times \mathbb{R}$ and solve the considered Cauchy problem. The solution is $t(\alpha) = -\frac{1}{2}\alpha + \tau_0$, $\alpha \in [0, \pi)$. Thus orthogonal trajectories to the isoptics to the circle C are half-lines

$$\gamma(\alpha) = F\left(\alpha, \tau_0 - \frac{1}{2}\alpha\right) = \frac{r}{\cos \frac{\alpha}{2}} e^{i\tau_0}, \quad \alpha \in [0, \pi),$$

starting from $z(\tau_0) = r e^{i\tau_0}$, where $\tau_0 \in [0, 2\pi)$.

2. For the ellipse $C: x^2 + \frac{y^2}{4} = 1$ the support function is $p(t) = \sqrt{\cos^2 t + 4 \sin^2 t}$, $t \in \mathbb{R}$. The function H is more complicated and we do not have an analytic solution of (10). Orthogonal trajectories to the isoptics of the ellipse obtained numerically in Wolfram Mathematica are presented in Fig. 3 a).
3. The graphs of orthogonal trajectories for isoptics of the curve with the support function

$$p(t) = \begin{cases} r, & t \in [0, \frac{\pi}{2}), \\ r - 17a + 20a \cos t + 20a \sin t - 16a \sin 2t - \\ \quad - 4 \cos 3t + 4a \sin 3t + a \cos 4t, & t \in [\frac{\pi}{2}, 2\pi), \end{cases}$$

where $r = 150$, $a = 1$ are presented in Fig. 3 b).

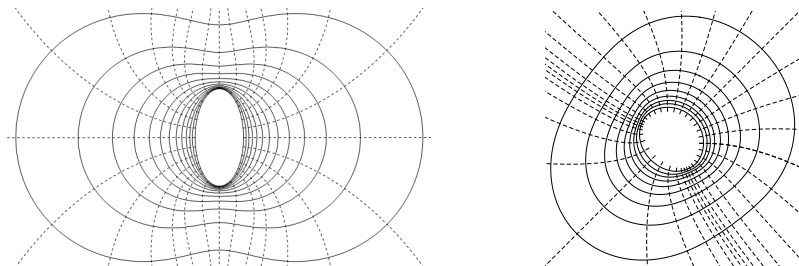


Fig. 3: Orthogonal trajectories of isoptics of a) the ellipse, b) the oval with the support function of class C^3

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Flexible cross-polytopes with two flat poses in three and four dimensions

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Abstract. According to R. Bricard 1897, there exist three types of flexible octahedra. The octahedra of type-3 are unsymmetric and admit two flat poses. The n -dimensional analogues of octahedra are called cross-polytopes. In 2014, A.A. Gaĭfullin determined with methods from algebraic geometry all flexible cross-polytopes in n -dimensional Euclidean, hyperbolic and spherical spaces for $n > 3$. The goal of this presentation is a synthetic approach to a particular family, the flexible 3- and 4-dimensional cross-polytopes with two flat poses. These poses are characterized by two geometric properties, local symmetries and concurrent axes of symmetry.

Keywords: flexible octahedron, flexible cross-polytope, flat pose.

1 Introduction

In 1897, Raoul Bricard [2] proved that in the 3-dimensional Euclidean space $\mathbb{E}^3$ there exist three types of flexible octahedra, i.e., polyhedra of the combinatorial type of a regular octahedron with rigid faces and hinges as edges while self-intersections are ignored. Type-3 octahedra are unsymmetric and admit two flat poses. Figure 1 shows how a flat pose can be constructed according to [1, Fig. 8] and also to [3, Fig. 297, p. 330].

The n -dimensional analogues of octahedra are called *cross-polytopes*. In 2014, Alexander A. Gaĭfullin [4] surprised the scientific community with the complete solution of a long-lasting open problem, namely the question for flexible cross-polytopes in the n -dimensional Euclidean, hyperbolic and spherical spaces for $n > 3$. Before that, only particular flexible examples in $\mathbb{E}^4$ were known (see [5]). Based on algebraic methods, Gaĭfullin succeeded to classify the flexible types, and he even presented parametrizations of the flexions in terms of Jacobian elliptic functions.

From the algebraic point of view, the n -dimensional analogues of Bricard's type-3 octahedra are the simplest. While Gaĭfullin showed their existence in an algebraic way, the goal of this presentation is to analyse the geometry of these flexible cross-polytopes in $\mathbb{E}^3$ and $\mathbb{E}^4$. Our novel approach starts from the existence of two flat poses.

Following the notation in [4], the cross-polytopes $\mathcal{C}^n$ in the Euclidean n -space $\mathbb{E}^n$ have $2n$ vertices coupled into pairs of *opposite* vertices (A_i, B_i) , $i = 1, \dots, n$. The 2^n hyperfaces (or *facets*) of $\mathcal{C}^n$ are the simplices $X_1 \dots X_n$ where X_i stands either for A_i or B_i . The $4 \binom{n}{2} = 2n(n-1)$ edges of $\mathcal{C}^n$ are $X_i X_j$ for $i \neq j$. More general, the k -faces, $1 < k \leq n-1$

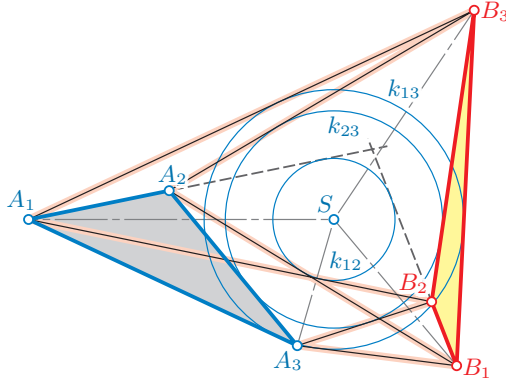


Fig. 1: Bricard's construction of a flat pose of a flexible type-3 octahedron: Two given concentric circles k_{12} , k_{31} and two points A_1 , B_1 define the remaining vertices $A_2, \dots, B_3$ and a third contacting circle k_{23} .

of $\mathcal{C}^n$ are the $2^{k+1} \binom{n}{k+1}$ simplices with $k+1$ vertices $X_{i_0} \dots X_{i_k}$ out of $\{A_1, \dots, A_n, B_1, \dots, B_n\}$ such that the indices $i_0 \dots i_k$ are mutually different. Hence, no k -face contains a pair (A_i, B_i) of opposite vertices. Moreover we assume that no k -face is of dimension smaller than k , i.e., flat.

According to [4], a polyhedron is called *flexible*, if we have hinges at all edges, and the polyhedron can deform while each hyperface remains congruent to itself. In other words, only the dihedral angles between neighboring facets can vary. We are interested in nontrivial deformations, which means that the deformation is not induced by a motion of the rigid cross-polytope in the ambient space.

When studying the flexibility of cross-polytopes $\mathcal{C}^n$, we still follow the convention in [4] and assume that the simplex $\Delta_0 := A_1, \dots, A_n$ is fixed while the opposite simplex $\Delta := B_1, \dots, B_n$ is moving. This means that the vertex B_i can rotate about the $(n-2)$ -dimensional *axis* spanned by $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ while tracing a circle. Point B_i rotates together with the simplex called *wing* $\Delta_i := A_1 \dots A_{i-1} B_i A_{i+1} \dots A_n$ relative to the fixed simplex Δ_0 (note the flexible octahedron in Figure 5 as an example for $n = 3$).

At flexible cross-polytopes there exist simultaneous rotations of all n wings $\Delta_1, \dots, \Delta_n$ such that all $\binom{n}{2}$ edges of the moving simplex Δ preserve their lengths $\overline{B_i B_j}$. We call such a self-movement of $\mathcal{C}^n$ a *flexion*.

Below we focus on cases, where during the flexion the cross-polytope passes through two incongruent flat poses where the moving simplex Δ is located in the hyperplane spanned by Δ_0 . When for each vertex $B_i \in \Delta$

the two corresponding positions B'_i and B''_i are different, we speak briefly of a *twice-flat* flexible cross-polytope.

2 Revisiting Bricard's type-3 octahedra

Let $A_1, B_1, A_2, B_2, A_3, B_3$ be the vertices of a three-dimensional cross-polytope $\mathcal{C}^3$.

2.1 Local symmetries of twice-flat octahedra

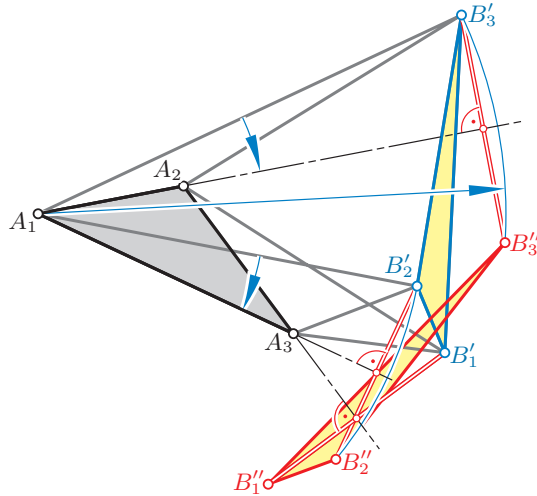


Fig. 2: At a twice-flat octahedron the moving triangle varies between $B'_1B'_2B'_3$ and $B''_1B''_2B''_3$. The double lines are projections of circular paths of the moving vertices in the case of a flexible octahedron.

Lemma 1. *A flat octahedron with the fixed triangle $A_1A_2A_3$ and the coplanar moving triangle $B'_1B'_2B'_3$ admits a second flat pose $B''_1B''_2B''_3$ with $B'_i \neq B''_i$ for all $i \in \{1, 2, 3\}$ if and only if at each vertex A_i the connecting lines with the remaining pairs (A_j, B'_j) of opposite vertices have common axes of symmetry.*

Proof. The half-rotation in $\mathbb{E}^3$ about the axis $[A_1, A_2]$ sends B'_3 to B''_3 , and that about $[A_1, A_3]$ sends B'_2 to B''_2 .¹ Thus, the bisectors of (B'_2, B''_2) and (B'_3, B''_3) intersect at A_1 , while there are equal distances $\overline{B'_2B'_3} = \overline{B''_2B''_3}$.

¹Throughout the paper we use $[X, Y, \dots, Z]$ as the symbol for the affine hull of the listed points, e.g., $[X, Y]$ for the *line* connecting the points X and Y , while the symbol $XY \dots Z$ stands for the simplex with vertices $X, Y, \dots, Z$, and in particular XY for the *segment* terminated by X and Y .

This implies that A_1 is the center of a 2D-rotation that brings $B'_2B'_3$ to $B''_2B''_3$. Half of the angle of this rotation is marked in Figure 2 and reveals that pairs of lines $([A_1, A_2], [A_1, B'_2])$ and $([A_1, B'_3], [A_1, A_3])$ share the angle bisectors.

Moreover, the flat pose shows that at the flexible four-sided pyramid connecting A_1 with the sides of the quadrangle $A_2A_3B_2B_3$ opposite apex angles are equal. We speak of an *isogonal pyramid*; its intersection with the unit sphere centered at A_1 gives a spherical isogram (Figure 4) with opposite sides of equal or complementary length. It is wellknown (see, e.g., [6, eq. (9)]) that during the flexion of this pyramid the tangents of the half dihedral angles φ_3 along A_1A_2 and φ_2 along A_1A_3 are proportional (note below Lemma 2). $\square$

The found symmetry at A_1 occurs similarly at A_2 and A_3 , and also at all B' -points since for the existence of two flat poses it is not relevant which face of the octahedron is fixed. We speak briefly of *local symmetries* at the flat pose. It will turn out that these are not sufficient for the flexibility of a twice-flat octahedron.

2.2 Transmission between adjacent wings

Let the points $A_1, \dots, B'_3$ of a flat pose satisfy the local symmetries. For obtaining an exact formula for the transmission between wings, we assume that the fixed triangle $A_1A_2A_3$ is counter-clockwise oriented, and the directions of the axes of rotations a_1, a_2, a_3 are given, respectively, by the directed segments A_2A_3 , A_3A_1 and A_1A_2 . The signed interior angles of $A_1A_2A_3$ are denoted by $\alpha_1, \alpha_2, \alpha_3$ (Figure 3).

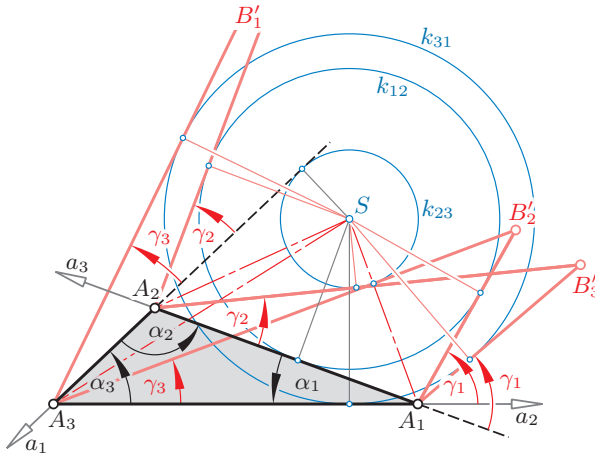


Fig. 3: The angles in the fixed face and the wings of a type-3 octahedron.

With γ_1 we denote the signed angle at A_1 between the lines $[A_1, A_3]$, $[A_1, B'_2]$. It equals the angle between $[A_1, A_2]$ and $[A_1, B'_3]$. We use the rotation by γ_1 about A_1 to transfer the orientation of a_2 to the line $[A_1, B'_2]$ and denote with p_1 the signed distance of $A_1 B'_2$, and similarly q_1 that of $A_1 B'_3$. An analogous procedure yields the signed angles γ_2, γ_3 and the signed distances p_2, q_2 and p_3, q_3 at the remaining vertices A_2 and A_3 . We assume for all i that $0 < \alpha_i, \gamma_i < \pi$ and $(\alpha_i + \gamma_i), (\alpha_i - \gamma_i) \neq k\pi$ for all $k \in \mathbb{Z}$.

Lemma 2. *Referring to the previous notation, the angles of rotation φ_2 of the wing $A_3 A_1 B'_2$ about a_2 and φ_3 of $A_2 A_1 B'_3$ about a_3 relative to $A_1 A_2 A_3$ with $t_i := \tan \frac{\varphi_i}{2}$ for $i = 2, 3$ are related by*

$$t_3 = \frac{-\sin \gamma_1 \pm \sin \alpha_1}{\sin(\gamma_1 + \alpha_1)} t_2.$$

The cases with the upper and the lower sign are respectively equivalent to

$$t_3 = \frac{\sin \frac{\alpha_1 - \gamma_1}{2}}{\sin \frac{\alpha_1 + \gamma_1}{2}} t_2 \quad \text{and} \quad t_3 = -\frac{\cos \frac{\alpha_1 - \gamma_1}{2}}{\cos \frac{\alpha_1 + \gamma_1}{2}} t_2.$$

Proof. We use a right-handed coordinate frame with the flat pose in the plane $z = 0$, with A_1 as origin and a_2 as positive x -axis. From points

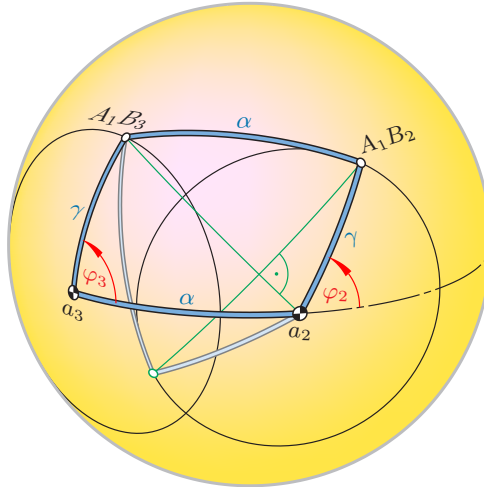


Fig. 4: The flexion of the isogonal pyramid with apex A_1 induces on spheres centered at A_1 two spherical motions with bifurcations at the aligned poses.

in the half-space $z > 0$ the triangle $A_1A_2A_3$ appears counter-clockwise oriented, as shown in Figure 3. For the sake of brevity, we temporarily suppress the subscript 1 at α_1 and γ_1 .

Thus, the circular paths of B_2 and B_3 (Figure 5) can be parametrized as

$$B_2 = p \begin{pmatrix} \cos \gamma \\ \sin \gamma \cos \varphi_2 \\ \sin \gamma \sin \varphi_2 \end{pmatrix}, \quad B_3 = q \begin{pmatrix} -\cos \alpha \cos \gamma - \sin \alpha \sin \gamma \cos \varphi_3 \\ \sin \alpha \cos \gamma - \cos \alpha \sin \gamma \cos \varphi_3 \\ \sin \gamma \sin \varphi_3 \end{pmatrix}.$$

The request that the distance $\overline{B_2B_3}$ remains equal to $\overline{B'_2B'_3}$ in the initial flat pose with $\varphi_2 = \varphi_3 = 0$ is equivalent to

$$\begin{aligned} & (p \cos \gamma + q \cos \alpha \cos \gamma + q \sin \alpha \sin \gamma \cos \varphi_3)^2 \\ & + (p \sin \gamma \cos \varphi_2 - q \sin \alpha \cos \gamma + q \cos \alpha \sin \gamma \cos \varphi_3)^2 \\ & + (p \sin \gamma \sin \varphi_2 - q \sin \gamma \sin \varphi_3)^2 = p^2 + q^2 + 2pq \cos \alpha. \end{aligned}$$

It implies

$$\begin{aligned} & \sin \alpha \sin \gamma \cos \gamma (\cos \varphi_3 - \cos \varphi_2) - \cos \alpha \sin^2 \gamma (1 - \cos \varphi_2 \cos \varphi_3) \\ & - \sin^2 \gamma \sin \varphi_2 \sin \varphi_3 = 0 \end{aligned}$$

and, after the substitution $t_i := \tan \frac{\varphi_i}{2}$ for $i = 2, 3$,

$$2 \sin \alpha \cos \gamma (t_2^2 - t_3^2) - 2 \cos \alpha \sin \gamma (t_2^2 + t_3^2) - 4 \sin \gamma t_2 t_3 = 0,$$

hence

$$\sin(\alpha - \gamma) t_2^2 - \sin(\alpha + \gamma) t_3^2 - 2 \sin \gamma t_2 t_3 = 0.$$

This reveals that in the particular case of isogonal pyramid the biquadratic relation (according to [6, eq. (4)] or [4, eq. (3.3)]) splits into two linear functions

$$t_3 = \frac{-\sin \gamma \pm \sqrt{\sin^2 \gamma + \sin^2 \alpha \cos^2 \gamma - \cos^2 \alpha \sin^2 \gamma}}{\sin(\alpha + \gamma)} t_2 = \frac{-\sin \gamma \pm \sin \alpha}{\sin(\alpha + \gamma)} t_2.$$

Thus we obtain either

$$t_3 = \frac{\sin \frac{\alpha - \gamma}{2}}{\sin \frac{\alpha + \gamma}{2}} t_2 \quad \text{or} \quad t_3 = -\frac{\cos \frac{\alpha - \gamma}{2}}{\cos \frac{\alpha + \gamma}{2}} t_2.$$

Both flat poses offer bifurcations between two rational movements. Their spherical images are depicted in Figure 4. $\square$

Lemma 3. *Let (A_i, B'_i) for $i = 1, 2, 3$ be coplanar pairs of points with local symmetries. Then, referring to Lemma 2, all points $S \neq A_1$ on one of the common axes of symmetry between the lines $[A_1, A_2]$, $[A_1, B'_2]$ and $[A_1, A_3]$, $[A_1, B'_2]$ satisfy the relation*

$$t_3 = \frac{d(S, a_2)}{d(S, a_3)} t_2,$$

where $d(S, a_i)$ denotes the signed distance of S to the revolute axis a_i such that for points in the interior of $A_1A_2A_3$ the distances to all axes are positive.

Proof. Let (i, j, k) by a cyclic permutation of $(1, 2, 3)$. If one symmetry axis through A_i meets the opposite side A_jA_k , then it includes with the axes a_k and a_j the angles $\frac{\alpha_i - \gamma_i}{2}$ and $\frac{\alpha_i + \gamma_i}{2}$, respectively (note vertex A_3 in Figure 3). Therefore holds, by virtue of Lemma 2,

$$\frac{-\sin \gamma_i + \sin \alpha_i}{\sin(\gamma_i + \alpha_i)} = \frac{\sin \frac{\alpha_i - \gamma_i}{2}}{\sin \frac{\alpha_i + \gamma_i}{2}} = \frac{d(S, a_j)}{d(S, a_k)}.$$

Otherwise (note points A_1 or A_2 in Figure 3) one of the axes includes the angles $\frac{\pi + \alpha_i - \gamma_i}{2}$ or $\frac{\pi - \alpha_i + \gamma_i}{2}$ and $\frac{\pi - \alpha_i - \gamma_i}{2}$ with a_j and a_k , and we obtain

$$\frac{-\sin \gamma_i - \sin \alpha_i}{\sin(\gamma_i + \alpha_i)} = -\frac{\cos \frac{\alpha_i - \gamma_i}{2}}{\cos \frac{\alpha_i + \gamma_i}{2}} = \frac{d(S, a_j)}{d(S, a_k)}.$$

This confirms the claim. The absolute values of the distances $d(S, a_i)$ are the radii of the concentric circles k_{12}, k_{23}, k_{31} in Figures 1 and 3. $\square$

2.3 Necessary and sufficient condition for flexibility

The following theorem rephrases a part of Bricard's main result in [2]. The new formulation paves the way to higher-dimensional versions.

Theorem 1. *Six coplanar points $A_1, \dots, B'_3$ define a twice-flat flexible octahedron in $\mathbb{E}^3$ if and only if at each vertex there holds local symmetry according to Lemma 1 and one of the axes of symmetry at each vertex passes through a common (finite or infinite) point S .*

Proof. For the sake of brevity we only prove that the condition is sufficient for finite S : By virtue of Lemma 3, all side lengths of the moving triangle are preserved, since for the transmissions between the wings' rotations from Δ_1 to Δ_2 , from Δ_2 to Δ_3 , and finally from Δ_3 to Δ_1 holds

$$\frac{d(S, a_3)}{d(S, a_1)} \cdot \frac{d(S, a_2)}{d(S, a_3)} \cdot \frac{d(S, a_1)}{d(S, a_2)} t_1 = t_1.$$

This confirms the flexibility. $\square$

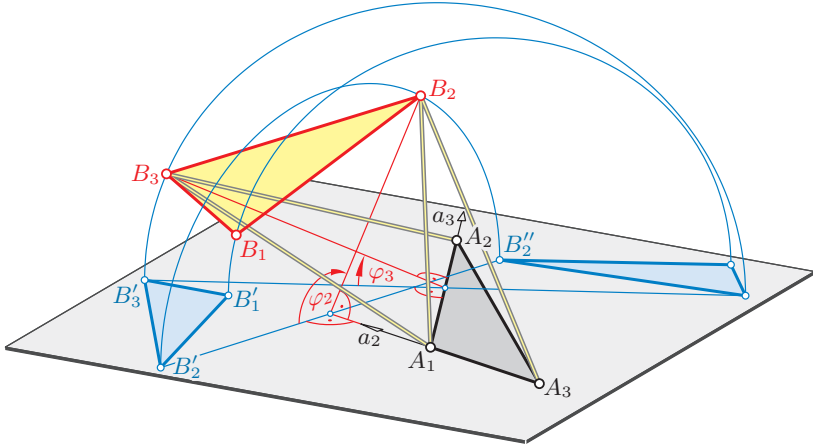


Fig. 5: Spatial pose of the moving triangle $B_1B_2B_3$ with the corresponding rotation angles φ_2, φ_3 of B_2, B_3 and wings $A_1A_3B_2$ and $A_1A_2B_3$ along with the respective axes a_2 and a_3 .

3 Outlook on twice-flat flexible cross-polytopes in $\mathbb{E}^4$

Four-dimensional cross-polytopes $\mathcal{C}^4$ contain four pairs of opposite vertices $(A_1, B_1), \dots, (A_4, B_4)$. Their 16 facets are tetrahedra $X_1X_2X_3X_4$ where X stands for A or B . They contain 24 edges X_iX_j , $i \neq j$, and 32 triangles $X_iX_jX_k$ with mutually different indices $i, j, k \in \{1, \dots, 4\}$.

We still follow Bricard [2] and Gaïfullin [4] and study the flexibility of $\mathcal{C}^4$ by keeping the tetrahedron $\Delta_0 := A_1, \dots, A_4$ fixed, while the opposite tetrahedron $\Delta := B_1, \dots, B_4$ moves. Vertex B_1 rotates together with the wing $\Delta_1 := A_2A_3A_4B_1$ about the plane $\varepsilon_1 = [A_2, A_3, A_4]$ and traces a circle in a plane orthogonal to ε_1 . The other vertices $B_2, B_3, B_4 \in \Delta$ move similarly.

We call $\mathcal{C}^4$ *twice-flat* if it admits two flat poses with the moving tetrahedron placed as $B'_1 \dots B'_n$ or $B''_1 \dots B''_n$ in the three-dimensional span of Δ_0 , where additionally holds $B'_i \neq B''_i$ for all $i \in \{1, \dots, 4\}$.

The cross-polytope $\mathcal{C}^4$ is flexible if the four wings $\Delta_1, \dots, \Delta_4$ can rotate continuously relative to Δ_0 in such a way that all six distances $\overline{B_iB_j}$ are preserved. During this motion all tetrahedra of $\mathcal{C}^4$ remain rigid. Only the dihedral angles between adjacent facets vary.

3.1 Local symmetries of twice-flat cross-polytopes

Suppose that the moving tetrahedron Δ of $\mathcal{C}^4$ admits two flat poses $B'_1 \dots B'_n$ and $B''_1 \dots B''_n$ with $B'_i \neq B''_i$ for all $i \in \{1, \dots, 4\}$. Then the second pose B''_1 of B_1 arises from the first pose B'_1 by a 4D-rotation through

180° about ε_1 , i.e., by a 3D-reflection in the plane ε_1 within the three-space spanned by Δ_0 . Consequently, the plane $\varepsilon_1 = [A_2, A_3, A_4]$ is the bisecting plane between B'_1 and B''_1 . Similarly, ε_2 is the bisecting plane between B'_2 and B''_2 . Since $\overline{B'_2 B'_3} = \overline{B''_2 B''_3}$, there exists a 3D-rotation about $[A_3, A_4] = \varepsilon_1 \cap \varepsilon_2$ with $B'_1 \mapsto B''_1$ and $B'_2 \mapsto B''_2$.

The rotation about $[A_3, A_4]$ through half angle takes $[A_3, A_4, B'_1]$ to $\varepsilon_1 = [A_3, A_4, A_2]$ and $[A_3, A_4, B'_2]$ to $\varepsilon_2 = [A_3, A_4, A_1]$. Thus, the signed dihedral angles of the wings $\Delta_1 = A_3 A_4 B_1 A_2$ and $\Delta_2 = A_3 A_4 B_2 A_1$ along the edge $A_3 A_4$ are either congruent or complementary which implies a local symmetry at $A_3 A_4$.

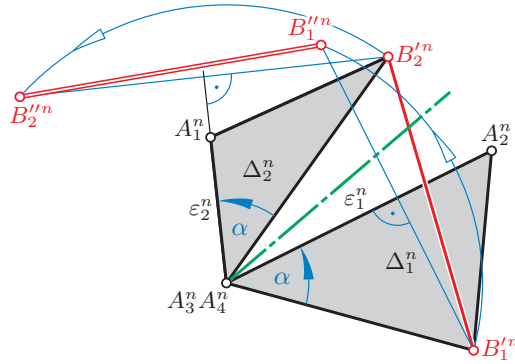


Fig. 6: Orthogonal view of $\mathcal{C}^4$ after projection parallel to the edge $A_3 A_4$.

Figure 6 shows this congruence after an orthogonal projection $X \mapsto X^n$ of $\mathbb{E}^4$ into a hyperplane Π orthogonal to $A_3 A_4$:

3.2 The transmission of rotations between adjacent wings

We study the transmission from the wing Δ_1 to Δ_2 , when the length of the moving edge $B_1 B_2$ remains constant. Δ_1 rotates with B_1 about $\varepsilon_1 = [A_2, A_3, A_4]$, while Δ_2 with B_2 rotates about $\varepsilon_2 = [A_1, A_3, A_4]$.

The two wings are connected by the facet $\Delta_{12} := A_3 A_4 B_1 B_2$ and by the fixed tetrahedron Δ_0 . The four tetrahedra Δ_1 , Δ_0 , Δ_2 , and Δ_{12} form a 4D-pyramid with the one-dimensional apex $[A_3, A_4]$. The orthogonal projection mentioned above sends it to a 3D-pyramid with the apex $A_3^n = A_4^n$ bounded by the quadrangle $B_1^n B_2^n A_1^n A_2^n$, and for each tetrahedron its dihedral angle along $A_3 A_4$ appears in true size. The flat pose of twice-flat cross-polytopes, as depicted in Figure 6, reveals that the 3D-pyramid is again isogonal. Moreover, orthogonal distances to ε_1 and ε_2 are shown in true size.

The dihedral angle φ_1 between the wing Δ_1 and Δ_0 is measured in a plane totally orthogonal to ε_1 . Since ε_1 contains $A_3 A_4$, the total-

orthogonal plane is parallel to the image-hyperplane Π of the orthogonal projection. Consequently, φ_1 appears in true size, and the same holds for $\varphi_2 = \angle \Delta_2 \Delta_0$ along ε_2 . Thus, the 4D-transmissions between adjacent wings satisfy formulas analogous to that derived in three-space², and we obtain, by virtue of Lemma 3, for the angles of rotation φ_i of the wings

$$t_j = \frac{d(S, \varepsilon_i)}{d(S, a_j)} t_i \quad \text{where} \quad t_k = \tan \frac{\varphi_k}{2}. \quad (1)$$

The symbol $d(S, \varepsilon_k)$ stands for the signed distance of S to the planes ε_k of the fixed tetrahedron Δ_0 such that for points in the interior of Δ_0 the distances to all faces are positive. Similar to Theorem 1 we conclude:

Theorem 2. *Eight points $A_1, \dots, B'_4$ in $\mathbb{E}^3$ define a flat pose of a twice-flat flexible cross-polytope C^4 in $\mathbb{E}^4$ if and only if at each edge of the fixed tetrahedron $A_1 \dots A_4$ there holds local symmetry between the connecting planes with the remaining pairs of opposite vertices, and one of the planes of symmetry passes through a common finite or infinite point S .*

4 Conclusion

A detailed kinematic analysis of Bricard's octahedra of type 3 paved the way to a characterization of those four-dimensional flexible cross-polytopes which admit two flat, i.e., three-dimensional poses. Further properties of this interesting family of higher-dimensional flexible polytopes will be disclosed in a further research.

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²This has already been mentioned in [4, Proposition 3.3].

Sundials and gnomonic azimuthal projection

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Abstract. We can look at sundials from the perspective of history, philosophy, astronomy, art, but also mathematics and geometry. Sundial dials contain images of the meridians and parallels of a spherical surface in a gnomonic azimuthal projection. Images of the meridians show hour lines, with the shadow of the pointer (called gnomon or style) showing the time during sunny days. Images of the parallels, here called declination lines, where the end of the shadow falls on certain significant days of the year, especially on the days of the equinox, summer and winter solstices. The aim of the paper is to show these connections, to formulate the constructions of the dial lines of several types of sundials using gnomonic azimuthal projection. The main contribution of this paper is in the creation of formulas for determining its parameters in relation to the geographical location of the dial plane, its orientation and the length of gnomon. In Wolfram Mathematica software, we demonstrate the rendering of the dials of various sundials using map equations and relations for determining the parameters of the gnomonic azimuthal projection.

Keywords: sundial, solar declination, gnomonic projection, gnomon

1 What we need to know about the Earth and the Sun

A sundial tells the time of day based on the apparent position of the Sun in the sky. Planar sundial dials contain hour and declination lines, they are images of the meridians and parallels of a spherical surface in a gnomonic azimuthal projection.

The construction of the sundial depends on the location of the installation site, which is expressed by coordinates on the reference surface of the Earth. Reference sphere of the Earth has a constant curvature at every point, its radius is R . Point M is given by **spherical geographic coordinates** (Fig.1):

- spherical latitude U : $U \in \langle -90^\circ, 90^\circ \rangle$.
- spherical longitude V : $V \in \langle -180^\circ, 180^\circ \rangle$.

Earth's parallel has constant latitude U and Earth's meridian has constant longitude V .

The position of the Sun in relation to the Earth is expressed by the solar declination (Fig.1). **Solar declination** δ is an angular distance of the line connecting the Sun with the center of the Earth from the equator, measured along the solar declination circle, positive toward the North Pole, negative toward the South Pole [7]. Solar declination δ is approximately the same during the day and the solar declination angles δ are in the interval from the Tropic of Cancer to the Tropic of Capricorn, hence between $\pm 23^\circ 26' 22''$.

Figure 3 shows a planar sundial and its components. The dial plate of a planar sundial is the plane π . The gnomon (so-called „style“ or „pointer“) $\mathbf{PS}$ casts a shadow $\mathbf{PS'}$ onto the dial, which indicates the hour line. The gnomon must be parallel to the axis of the Earth's rotation for the sundial to be accurate throughout the year. The shadow of the gnomon's endpoint during the day describes the declination (datum) line which correspond to declination angle δ on certain significant days of the year, especially on the days of the equinox, summer and winter solstices and when the Sun enters the individual "signs" of the zodiac [5].

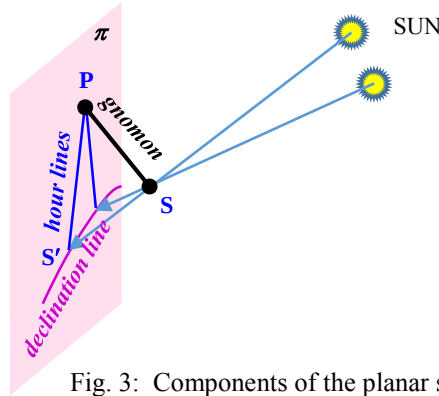


Fig. 3: Components of the planar sundial

2 Gnomonic azimuthal projection

The gnomonic azimuthal projection is the central perspective projection of a reference sphere Γ onto a plane, while the center projection S equals the center of reference sphere Γ (Fig. 4) [4].

The author of the gnomonic projection is the Greek astronomer, mathematician, and philosopher Thales of Miletus (624-547 BC).

The following properties apply in the gnomonic azimuthal projection (Fig. 4):

- 1st The gnomonic images of the two extreme points of the diameter of the reference sphere are identical.
- 2nd The gnomonic image of all the great circles of the sphere Γ are straight lines.

From the above properties, other properties follow:

- 3rd The gnomonic projection of all meridians, the equator, and the orthodromes of the reference sphere Γ are straight lines.
- 4th The gnomonic images of the geographic poles are identical (real point or point at infinity).
- 5th Gnomonic images of meridians are straight lines passing through the image of the poles (a pencil of lines, parallel lines).
- 6th Gnomonic images of geographic parallels are conic sections, namely ellipses, parabolas, hyperbolas.

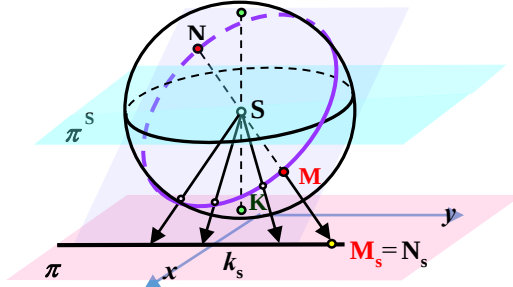


Fig. 4: The principle and properties of gnomonic azimuthal projection

We know three types of gnomonic projection by position of the projection plane:

- **Normal (polar) aspect** – projection plane π is perpendicular to the Earth's axis.
- **Transversal aspect** – projection plane π is perpendicular to the equatorial plane.
- **Oblique aspect** – projection plane π is not perpendicular to the Earth's axis, nor the equatorial plane.

The geographical coordinates of the tangent point **K** of the reference spherical surface with the projection plane (the so-called cartographic pole and the origin of the coordinate system) are U_K and V_K . Radius of spherical surface is R and the **map equations of gnomonic azimuthal projection** are:

$$\begin{aligned} x &= \frac{R \sin U \cos U_K - \cos U \sin U_K \cos(V - V_K)}{\sin U \sin U_K + \cos U \cos U_K \cos(V - V_K)}, \\ y &= \frac{R \cos U \sin(V - V_K)}{\sin U \sin U_K + \cos U \cos U_K \cos(V - V_K)}, \end{aligned} \quad (1)$$

3 Gnomonic azimuthal projection on the sundials

The use of gnomonic projection on sundials has been discussed, but we lacked a simple explanation of why and how. Therefore, we created our own illustration and explanation of relation between sundial and gnomonic azimuthal projection (Fig. 5). In the chapter 1, we mentioned, that the position of the sundial plate and the angle between the gnomon and the plate depend on the geographical coordinates U_L and V_L of the installation location. The gnomon **PS** must be parallel to the axis of the Earth's rotation. The planar sundial dials contain images of the meridians and parallels of a spherical surface in a gnomonic azimuthal projection. **Now we will define this spherical surface Γ .**

Let the solar declination be δ , then on this day the Sun's rays have this direction and pass through point **S**. Since the declination line is the set of shadows

of the endpoint S of the gnomon, let us construct a spherical surface Γ with center in the endpoint S of the gnomon and Γ is tangent to the dial plane π . The spherical surface Γ is homothetic to the reference sphere of the Earth (Fig. 5).

Due to the Earth's rotation, the declination line is the image of a parallel of the spherical surface Γ , where the spherical latitude is the declination δ , in gnomonic azimuthal projection with center S onto a plane π .

The gnomon lies on the axis of rotation of the spherical surface Γ , so the shadows of the gnomon, and therefore the hour lines, are the gnomonic projection of the meridians. The images of meridians are straight lines and the image of the local meridian shows noon, i.e. 12 o'clock.

Image of the equator of this spherical surface Γ is the straight line. It is the path of the shadow of the end point of the gnomon during the equinoxes, when the Sun is on the equatorial plane.

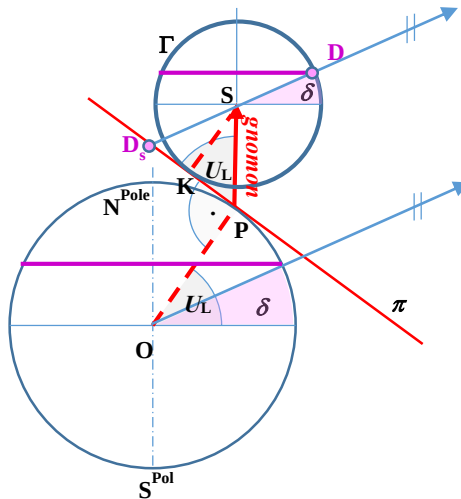


Fig. 5: The principle of using gnomonic azimuthal projection on sundials

3.1 Geometric construction of hour lines and declination lines on horizontal and vertical direct south sundials

The lines on the dial of horizontal and vertical sundials are images of meridians and parallels in gnomonic projection in oblique aspect (Fig. 6). The projection plane is tangent to the spherical surface at point K whose spherical coordinates are related to the spherical coordinates U_L , V_L of the sundial's location.

The image of the equator is a straight line. The base of the gnomon is an image of the Earth's pole. The image of the local meridian with the longitude V_L shows 12 o'clock. The other hour lines are identical to the images of the meridians, whose the angular spacing is in integer multiples hour angle 15° from the local meridian. The images of the meridians are straight lines passing through

3.3 Derivation of the parameters of the gnomonic projection on a declining vertical sundials defined by azimuth A

The lines corresponding to specific dates, known as datum or declination lines, are constructed as gnomonic images of the parallels on the spherical surface. To achieve this, we first construct an auxiliary view for the gnomonic azimuthal projection of this spherical surface onto the dial plane π .

At point $\mathbf{H''}$ we choose the image of the projection plane π . We derive the center $\mathbf{S_2}$ of the spherical surface from its rotated position $\mathbf{S_0}$. The angle between the Earth's axis and the projection plane is $90^\circ - U_L$, where U_L is the **local latitude**. The tangential point $\mathbf{K}$ (also called cartographic pole) has spherical latitude of $U_K = 90^\circ - U_L$. As we mentioned in chapter 3.1, in our latitudes, the images of parallels are hyperbolas with a common axis perpendicular to the image of the equator. We determine these hyperbolas using their principal vertices ${}^h\mathbf{A_s}$, ${}^h\mathbf{B_s}$ and focus ${}^h\mathbf{F}$.

When using the map equations of the gnomonic azimuthal projection, we also need the spherical longitude V_K of the cartographic pole $\mathbf{K}$ and the radius R of the spherical surface. The spherical longitude V_L of the local meridian indicates noon, so its gnomonic image passes through point $\mathbf{E}$ on the equator. The gnomonic projection of the meridian of the cartographic pole $\mathbf{K}$ with longitude V_K passes through point $\mathbf{S_0}$, therefore: $V_K = V_L - \angle(\mathbf{ES_0E'})$.

We derive the angle $\varepsilon = \angle(\mathbf{ES_0E'})$ as follows:

$$|\mathbf{E'S_0}| = R \sin U_L,$$

$$|\mathbf{PE'}| = |\mathbf{H''K}| + |\mathbf{KP'}| = R (\cot U_L + \tan U_L).$$

Considering $\angle(\mathbf{EPE'}) = \omega$ then after substitution:

$$|\mathbf{EE'}| = |\mathbf{PE'}| \tan \omega = R \tan \omega (\cot U_L + \tan U_L).$$

We formulate this angle ε :

$$\tan \varepsilon = \frac{|\mathbf{EE'}|}{|\mathbf{E'S_0}|} = \frac{R \tan \omega (\tan U_L + \cot U_L)}{R \sin U_L},$$

and after editing:

$$\varepsilon = \arctan \left(\frac{\tan \omega}{\sin^2 U_L \cos U_L} \right),$$

therefore:

$$V_K = V_L - \arctan \left(\frac{\tan \omega}{\sin^2 U_L \cos U_L} \right). \quad (3)$$

From the previously mentioned properties, we can derive the radius R of the spherical surface, depending on the length d of the gnomon. The position of the equator image depends on the length d of the gnomon, the local spherical latitude U_L , and the angle ω , which is also determined from the azimuth A of the wall. The length of the line segment $\mathbf{PE}$ in the triangle $\Delta\mathbf{PEH'}$ is:

$$|\mathbf{PE}| = \frac{d}{\sin U_L}.$$

We determine the length of the segment $\mathbf{PE}'$ in the triangle $\Delta\mathbf{PEE}'$, since $\angle(\mathbf{EPE}') = \omega$, then:

$$|\mathbf{PE}'| = |\mathbf{PE}| \cos \omega = d \frac{\cos \omega}{\sin U_L}. \quad (4)$$

In another way, the length of the line segment $\mathbf{PE}'$ can be determined from the size and position of the spherical surface, namely from its radius R and local spherical latitude U , so the following holds:

$$|\mathbf{PE}'| = |\mathbf{H''K}| + |\mathbf{KP}'| = R (\cot U_L + \tan U_L). \quad (5)$$

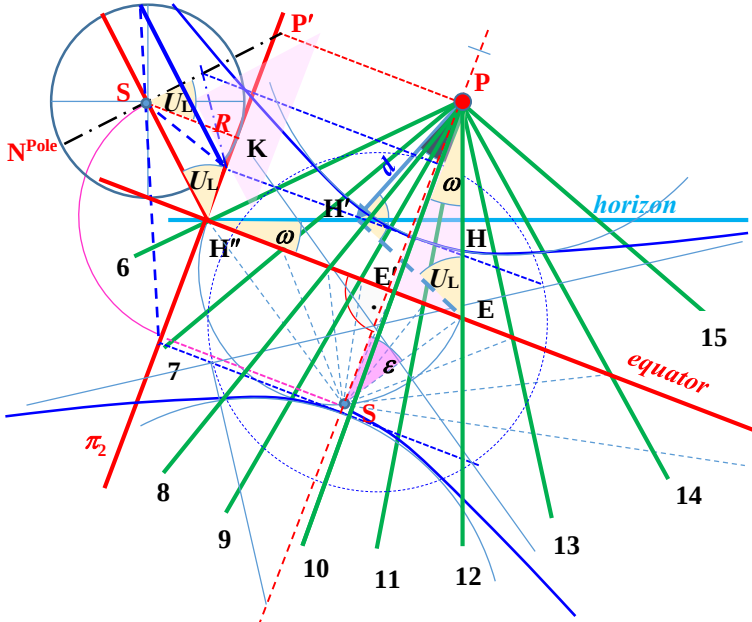


Fig. 8: Geometric construction of declination lines on declining vertical sundials defined by azimuth A

Equating the right sides of the relations (4) and (5), we obtain an expression for the radius R of the spherical surface:

$$R = \frac{d \frac{\cos \omega}{\sin U_L}}{\tan U_L + \cot U_L}, \text{ after modification: } R = d \cos \omega \cos U_L. \quad (6)$$

We can also reformulate this relationship:

$$R = d \cos \omega \cos (90^\circ - U_K). \quad (7)$$

4 Application of derived parameters to create a sundial dial

The parameters of the gnomonic projection, namely the spherical latitude and longitude U_K , V_K of the cartographic pole K and the radius R of the spherical surface, derived in the previous chapter can be applied to the creation of the dial of horizontal sundials, vertical direct south sundials, and also a declining vertical sundials defined by azimuth A . Considering the comparison of different types of sundials, we have created designs of dials for a specific location in **Bratislava** with the following spherical coordinates:

$$U_L = 48^\circ 8' 53.38'',$$

$$V_L = 17^\circ 6' 24.26''.$$

4.1 Horizontal sundials

The dial plate of horizontal sundial is parallel to the local tangent plane of the reference surface and the gnomon is parallel with Earth's axis and the angle between the gnomon and dial plate is equal to the local spherical latitude U_L .

The projection plane of gnomonic azimuthal projection is tangent to the spherical surface at the cartographic pole K , whose spherical coordinates are:

$$\text{spherical latitude } U_K = U_L,$$

$$\text{spherical longitude } V_K = V_L, \quad (8)$$

where U_L and V_L are the geographic coordinates of the sundials's installation location.

Since in equation (7) for calculating the radius R of the spherical surface, $\omega = 0^\circ$, then the following applies to horizontal sundials:

$$R = d \cos(90^\circ - U_L), \quad (9)$$

where d is length of gnomon.

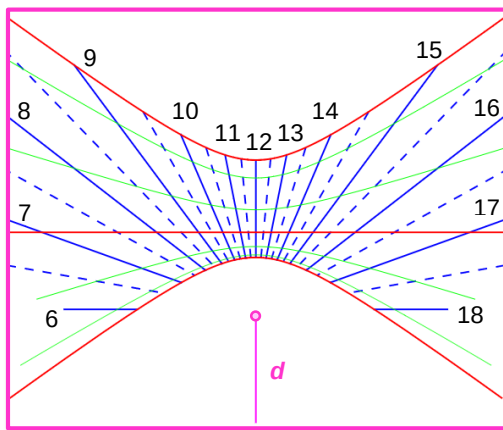


Fig. 9: Hour lines and declination lines in horizontal sundial for the Bratislava location

Figure 9 shows a visualization of hour and declination lines on horizontal sundial for the Bratislava location, which were created in the software Wolfram Mathematica [8] using the map equations (1) of the gnomonic azimuthal projection. The parameters U_K , V_K , R of gnomonic azimuthal projection in oblique aspect were determined using the relations (8) and (9), from spherical coordinates U_L , V_L of location Bratislava and from length of gnomon d , which is shown in the scale of the dial.

Figure 10 shows examples of horizontal sundials, on the left is sundial located in the center of the village Babiná, made in 2010 by folk craftsman M. Baran from Sliač and sundial on the right is located in the park near the Cultural Center in Bernolákovo. It was made according to the design of Mr. Milan Baran in 2011.



Fig. 10: Examples of the horizontal sundials located in Babiná (left) and in Bernolákovo (right) (photo by Ladislav Barabás) [3]

4.2 Vertical direct south sundials

Planar plate of vertical sundial is in normal plane to the reference surface, the gnomon is parallel to the Earth's axis, then the angle between the gnomon and the dial plate is: $90^\circ - U_L$, where U_L is the spherical latitude of the installation site. Vertical direct south sundials have dial plane perpendicular to the tangent to the south-facing meridian, its azimuth is $A = 0^\circ$.

The projection plane of gnomonic azimuthal projection in the oblique aspect is tangent to the spherical surface at the cartographic pole K , whose coordinates are:

$$\begin{aligned} \text{spherical latitude } U_K &= 90^\circ - U_L, \\ \text{spherical longitude } V_K &= V_L, \end{aligned} \quad (10)$$

where U_L and V_L are the geographic coordinates of the sundials's installation location.

Since in equation (6) for calculating the radius R of the spherical surface, $\omega = 0^\circ$, then the following applies to vertical direct south sundials:

$$R = d \cos U_L, \quad (11)$$

where d is length of gnomon.

Figure 11 shows a visualization of hour and declination lines on vertical direct south sundial for the Bratislava location, which were created in the software

Wolfram Mathematica [8] using the map equations (1) of the gnomonic azimuthal projection. The parameters U_K , V_K , R of gnomonic azimuthal projection in oblique aspect were determined using the relations (10) and (11), from spherical coordinates U_L , V_L of location Bratislava and from length of gnomon d , which is shown in the scale of the dial.

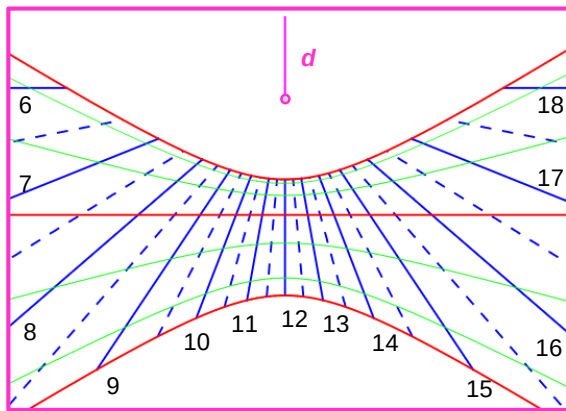


Fig. 11: Hour lines and declination lines in vertical direct south sundial for the Bratislava location



Fig. 12: Examples of vertical direct south, polar and horizontal sundials

Figure 12 shows examples of sundials. On the left is dial of sundial from ancient Egypt, Valley of the Kings (1500 BC) [6] and on the center and the right is vertical, polar and horizontal sundials in the of Brno Observatory and Planetarium area (photo by Vajsábllová).

4.3 Declining vertical sundials defined by azimuth A

Dial plane of declining vertical sundial defined by azimuth A is in normal plane to the reference surface with angle A to the tangent to the south plane. The gnomon is also parallel to the Earth's axis, then the angle between the gnomon

and the dial plate is: $90^\circ - U_L$, where U_L is the spherical latitude of the installation site.

The projection plane of gnomonic azimuthal projection in the oblique aspect is tangent to the spherical surface at the cartographic pole $\mathbf{K}$ with coordinates:

spherical latitude $U_K = 90^\circ - U_L$,

spherical longitude V_K is defined in (3):

$$V_K = V_L - \arctan\left(\frac{\tan \omega}{\sin^2 U_L \cos U_L}\right).$$

where U_L and V_L are the geographic coordinates of the sundials's installation location and declination ω is defined by (2):

$$\omega = A \frac{90^\circ - U_L}{90^\circ}.$$

Radius R of the spherical surface is defined by (6):

$$R = d \cos \omega \cos U_L.$$

where d is length of gnomon.

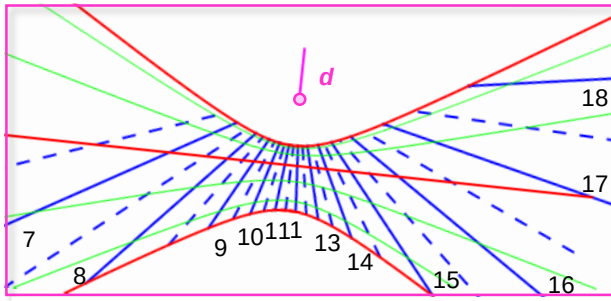


Fig. 13: Hour lines and declination lines in declining vertical sundial defined by azimuth $A = 10^\circ$ for the Bratislava location

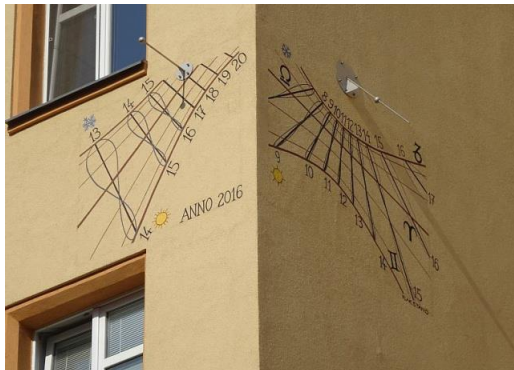


Fig. 14: Declining vertical sundials on two walls with different azimuth A in Bratislava, Bezručova [3]

Figure 13 shows a visualization of hour and declination lines on declining vertical sundial defined by azimuth $A = 10^\circ$ for the Bratislava location, which were created in the software Wolfram Mathematica using the map equations (1) of the gnomonic azimuthal projection. The parameters U_K , V_K , R of gnomonic azimuthal projection in oblique aspect were determined using the relations (3) and (6), from spherical coordinates U_L , V_L of location Bratislava, azimuth A and from length of gnomon d , which is shown in the scale of the dial. The image of the equator is a straight line inclined from the horizon by an angle ω , which is determined according to relation (2).

Figure 14 shows two declining vertical sundial defined by azimuth A .

5 Conclusion

Sundials are not just history. In 2022, the Technical Museum in Košice [3] published an overview of 245 sundials in Slovakia, an increase of 117 sundials since 1977.

In the paper, we showed a geometric solution to the dial design, both constructive and analytical, with a strict connection to the gnomonic azimuthal projection. Another benefit is the derivation of the parameters of the gnomonic azimuthal projection with application to several types of sundials. We applied these parameters when using the map equations of the gnomonic azimuthal projection in the creating a program for rendering a sundial dial in the Wolfram Mathematica software environment.

Acknowledgements

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- [8] WOLFRAM Mathematica 13.0. Copyright 1988-2020 Wolfram Research, Inc.

Pairs of inverse curves

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Abstract. This paper brings examples of curves that are inverse to some elementary planar curves in the complex plane, through algorithm based on point-wise operation of Minkowski division of point sets defined in the sense of Minkowski geometric algebra of complex sets. Pairs of inverse curves are presented for line, circle, conic sections, spirals and helix, as modelled in the GeoGebra software environment.

Keywords: complex sets, geometric algebra, division of points, inverse curve

1 Introduction

Minkowski geometric algebra of complex sets was introduced in 2001 by Rida T. Farouki and Bahram Ravani and published in paper [1]. Arithmetic operations of summation, difference, multiplication and division of points in the complex plane are defined in this paper as standard operations on complex numbers. Point-wise operations of Minkowski sum, difference, product and quotient of point sets in the complex plane are then defined as follows.

Let A and B be point sets in the complex plane. Then

Minkowski sum be set $A \oplus B = \{\mathbf{a} + \mathbf{b}; \mathbf{a} \in A \wedge \mathbf{b} \in B\}$

Minkowski difference be set $A \ominus B = \{\mathbf{a} - \mathbf{b}; \mathbf{a} \in A \wedge \mathbf{b} \in B\}$

Minkowski product be set $A \otimes B = \{\mathbf{a} \times \mathbf{b}; \mathbf{a} \in A \wedge \mathbf{b} \in B\}$

Minkowski quotient be set $A \oslash B = \{\mathbf{a} \div \mathbf{b}; \mathbf{a} \in A \wedge \mathbf{b} \in B, \mathbf{b} \neq (0,0)\}$

where $+$, $-$, $\times$ and $\div$ are standard operations on complex numbers. Let points $\mathbf{a}$ and $\mathbf{b}$ be determined in the algebraic form as $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2)$$

$$\mathbf{a} \times \mathbf{b} = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

$$\mathbf{a} \div \mathbf{b} = \frac{(a_1 b_1 + a_2 b_2, a_2 b_1 - a_1 b_2)}{b_1^2 + b_2^2}$$

Minkowski division of point sets A and B might be also regarded as Minkowski multiplication of set A by the inverse set B^{-1} to the set B . In order to

determine an inverse set B^{-1} to the set B , the following equation should be satisfied

$$B \times B^{-1} = B^{-1} \times B = \{(1,0)\}$$

yielding to the form of the inverse set determination as $B^{-1} = \frac{\{(1,0)\}}{B}$, whereas for points in the set B^{-1} this leads to the form $\mathbf{b}^{-1} = \frac{(1,0)}{\mathbf{b}} = \frac{b}{|\mathbf{b}|^2}$.

2 Inverse curves

Let us consider smooth curves in the complex plane as point sets to be multiplied and divided following the above rules, and let us look at the formula of an inverse curve to an arbitrary planar curve. Let the curve A be defined by its vector parameterisation $A: \mathbf{r}(u) = (x(u), y(u)), u \in I \subset \mathbf{R}$. Then, using the operation of Minkowski division of point sets, we receive

$$\mathbf{r}^{-1}(u) = \frac{\mathbf{1}}{\mathbf{r}(u)} = \frac{(1,0)}{\mathbf{r}(u)} = \frac{\mathbf{r}(u)}{|\mathbf{r}(u)|^2} = \frac{(x(u), y(u))}{(x(u))^2 + (y(u))^2}.$$

Looking for pairs of inverse curves, we will start with line, then continue with circle, ellipse, parabola and hyperbola, spiral and helix. We will investigate the influence of various positions of the basic curve on the form of its inverse.

2.1 Inverse to a line

Parametric representation of a line in the form

$$\mathbf{r}(u) = (u, ku + h), u \in \mathbf{R}, k, h \in \mathbf{R}$$

determines the form of the parameterisation of the line inverse

$$\mathbf{r}^{-1}(u) = \frac{(u, ku + h)}{u^2(k^2 + 1) + 2khu + h^2}, u \in \mathbf{R}, k, h \in \mathbf{R}.$$

Thus inverse to a line not passing through the origin is a circle, while the inverse is a line for a line passing through the origin, as illustrated in Fig. 1.

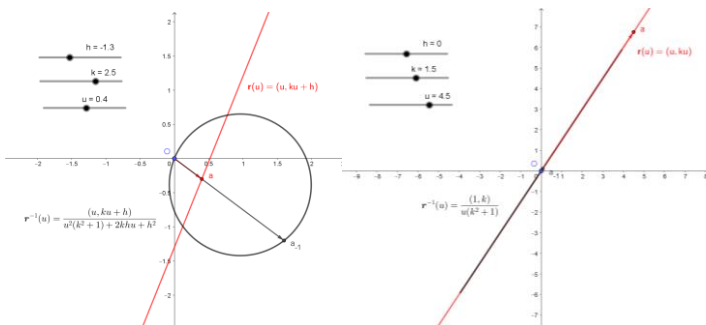


Fig. 1: Inverse curves to lines

2.2 Inverse to a circle

Considering parameterisation of a circle given by vector function

$$\mathbf{r}(u) = (k \cos u + d_1, k \sin u + d_2), u \in]0, 2\pi[, k, d_1, d_2 \in \mathbf{R}$$

there might be derived vector representation of its inverse curve in the form

$$\mathbf{r}^{-1}(u) = \frac{(k \cos u + d_1, k \sin u + d_2)}{k^2 + 2k(d_1 \cos u + d_2 \sin u) + d_1^2 + d_2^2}, u \in]0, 2\pi[, k, d_1, d_2 \in \mathbf{R}$$

With respect to the position of the circle, its inverse might be:

a line, if circle is passing through the origin

a circle, in all other positions, see in Fig. 2.

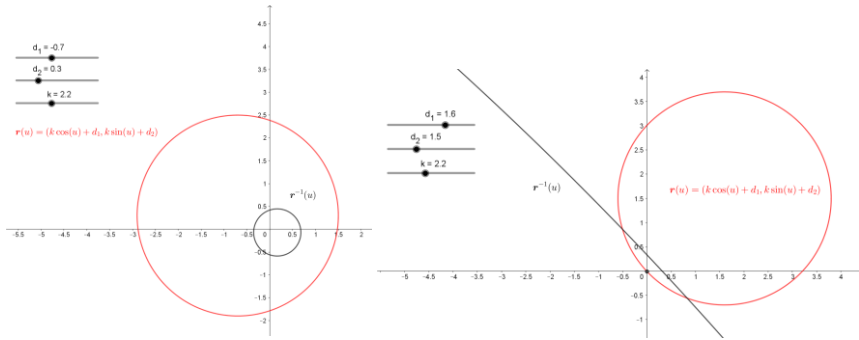


Fig. 2: Inverse curves to circles

2.3 Inverse to an ellipse

Inverse curve to an ellipse represented parametrically by vector map

$$\mathbf{r}(u) = (a \cos u + d_1, b \sin u + d_2), u \in]0, 2\pi[, a, b, d_1, d_2 \in \mathbf{R}$$

is determined by vector function

$$\mathbf{r}^{-1}(u) = \frac{(a \cos u + d_1, b \sin u + d_2)}{h(u)}, u \in]0, 2\pi[, a, b, d_1, d_2 \in \mathbf{R},$$

where $h(u) = a^2 \cos^2 u + b^2 \sin^2 u + 2(ad_1 \cos u + bd_2 \sin u) + d_1^2 + d_2^2$.

Inverse curve might have many different forms, depending on the position of ellipse. Inverse to ellipse with centre in the origin and axes in the coordinate axes resemble curves called hippopede of Eudoxus, if the ellipse is passing through the origin, its inverse curve looks like cissoid of Diocles. There are also many other forms of inverse curve to an ellipse, one of which might be perhaps also epitrochoid, in case of certain values of parameters describing the ellipse. Many other forms might be achieved by general position of ellipse with axes not coinciding with the coordinate axes and arbitrary centre in the plane, see Fig. 3.

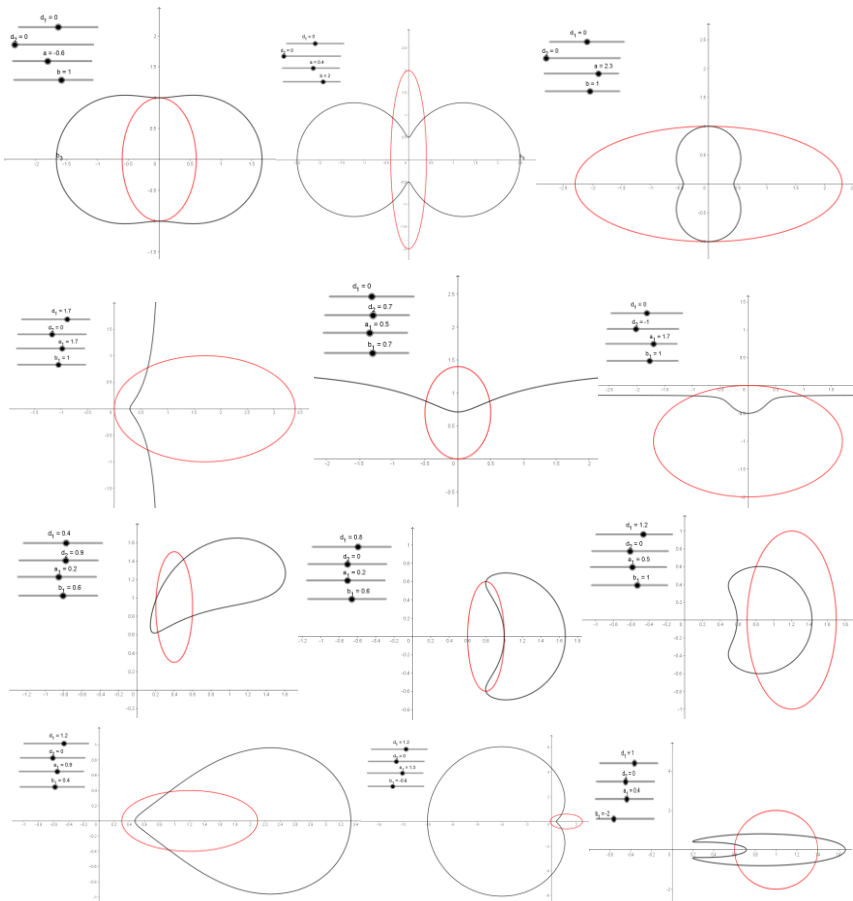


Fig. 3: Inverse curves to ellipses

2.4 Inverse to a parabola

Parabola in the basic position in the plane with axis coinciding or parallel to the coordinate axis y might be represented by vector function

$$\mathbf{r}(u) = (u, ku^2 + h), u \in \mathbf{R}, k, h \in \mathbf{R}$$

and its inverse curve has parametric representation in the form

$$\mathbf{r}^{-1}(u) = \frac{(u, ku^2 + h)}{u^2(1 + k^2u^2 + 2kh) + h^2}, u \in \mathbf{R}, k, h \in \mathbf{R}.$$

Inverse curves have the form of hippopede of Eudoxus, cissoid of Diocles, or even and epitrochoid, with respect to the position of parabola to the coordinate axes and the origin as illustrated in Fig. 4.

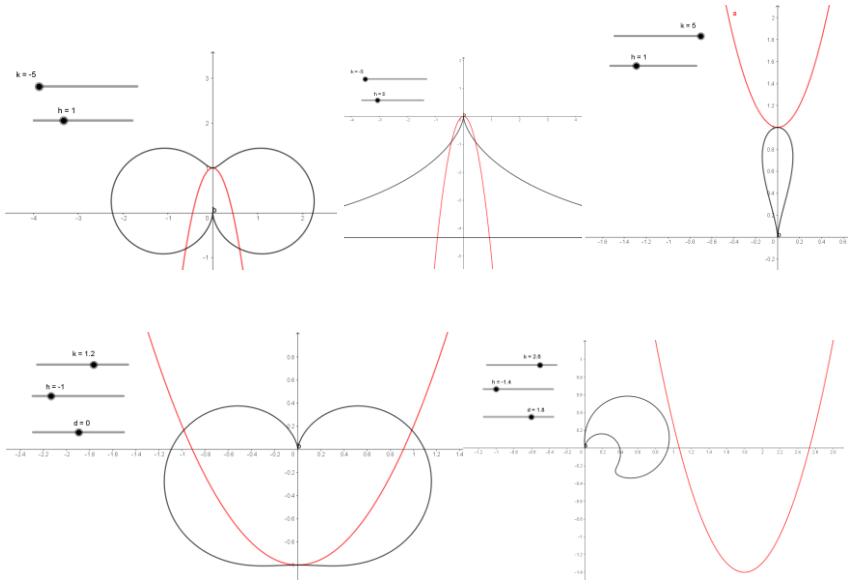


Fig. 4: Inverse curves to parabolas

2.5 Inverse to a hyperbola

Parametric representation of hyperbola in the form

$$\mathbf{r}(u) = \left(\frac{a}{c(u)} + d_1, b t(u) + d_2 \right), u \in]0, 1[, a, b, d_1, d_2 \in \mathbf{R}$$

for $c(u) = \cos((1 - 2u)\pi)$, $t(u) = \tan((1 - 2u)\pi)$ yields parametric representation of its inverse curve

$$\mathbf{r}^{-1}(u) = \frac{\mathbf{r}(u)}{h(u)}, u \in]0, 1[, a, b, d_1, d_2 \in \mathbf{R},$$

where $h(u) = a^2 c^{-2}(u) + b^2 t^2(u) + 2(ad_1 c^{-1}(u) + bd_2 t(u) + d_1^2 + d_2^2)$.

Lemniscate of Bernoulli is inverse curve to equiaxed hyperbola with equal semi-axes $a = b$ and axes in the coordinate axis, limaçon of Pascal (epitrochoid) is inverse to properly shifted equiaxed hyperbola, the inverse curve might be also Cartesian oval or even cissoids of Diocles if the equiaxed hyperbola is passing through the origin, as this was the case also of other conic sections, ellipse and parabola.

These specific forms of hyperbola inverse curves, and also some other interesting forms of hyperbola inverses, are presented in Fig. 5.

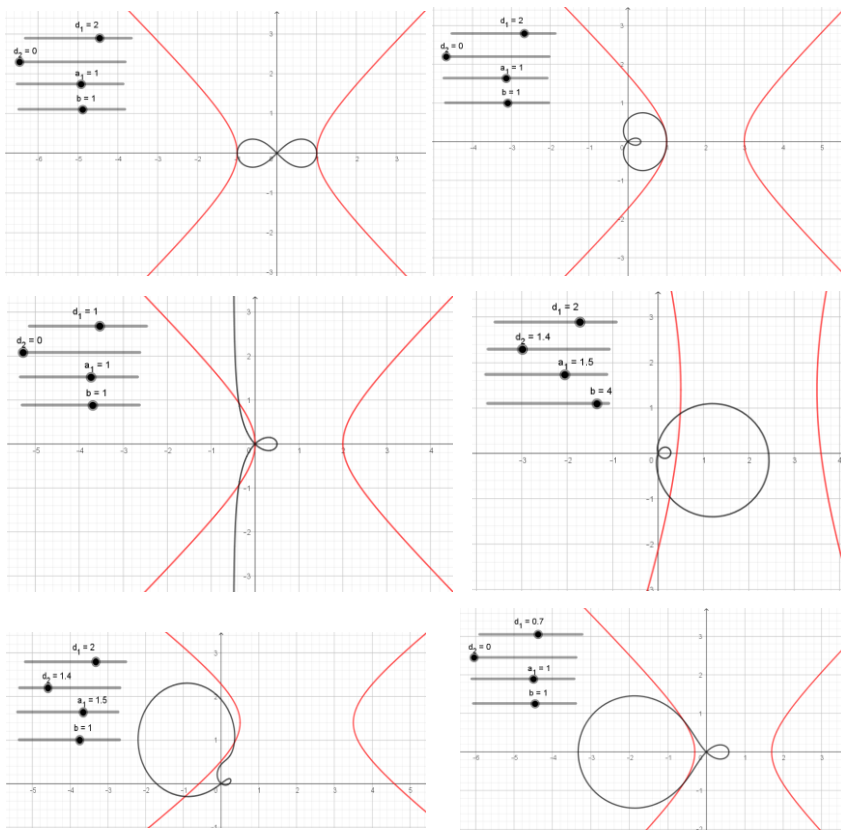


Fig. 5: Inverse curves to hyperbolas

2.6 Inverse to a spiral

Inverse curve to spiral

$$\mathbf{r}(u) = (ku \cos u + d_1, ku \sin u + d_2), u \in]0, 2\pi[, k, d_1, d_2 \in \mathbf{R}$$

is parameterised in the form

$$\mathbf{r}^{-1}(u) = \frac{(ku \cos u + d_1, ku \sin u + d_2)}{h(u)}, u \in]0, 2\pi[, k, d_1, d_2 \in \mathbf{R}$$

where $h(u) = k^2 u^2 + 2ku(d_1 \cos u + d_2 \sin u) + d_1^2 + d_2^2$.

Various interesting forms of inverse curves to spiral in several specific positions with respect to coordinate system, determined by presented chosen values of parameters k , d_1 , and d_2 , are illustrated in Fig. 6.

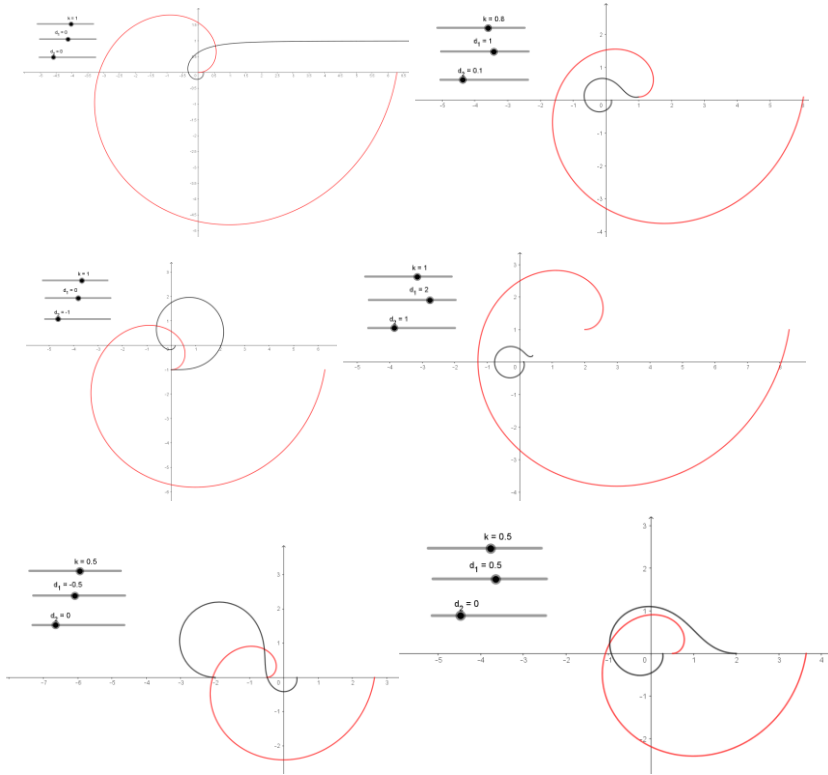


Fig. 6: Inverse curves to spirals

2.7 Inverse to a helix

Applying the same formula for definition of an inverse curve to a space curve determined parametrically by formula

$$\mathbf{r}(u) = (x(u), y(u), z(u)), u \in I \subset \mathbf{R},$$

there might be derived parametric form of such inverse as

$$\mathbf{r}^{-1}(u) = \frac{\mathbf{1}}{\mathbf{r}(u)} = \frac{(x(u), y(u), z(u))}{(x(u))^2 + (y(u))^2 + (z(u))^2}, u \in I \subset \mathbf{R}.$$

Considering helix in the form

$$\mathbf{r}(u) = (k \cos u + d_1, k \sin u + d_2, vu), u \in]0, 2\pi[, k, d_1, d_2, v \in \mathbf{R},$$

its inverse curve is defined by formula

$$\mathbf{r}^{-1}(u) = \frac{(k \cos u + d_1, k \sin u + d_2, vu)}{h(u)}, k, d_1, d_2, v \in \mathbf{R},$$

while $h(u) = k^2 + 2k(d_1 \cos u + d_2 \sin u) + d_1^2 + d_2^2 + v^2 u^2$.

Inverse curve to a helix in the basic position is another helix, generally, it is a space curve, see Fig. 7.

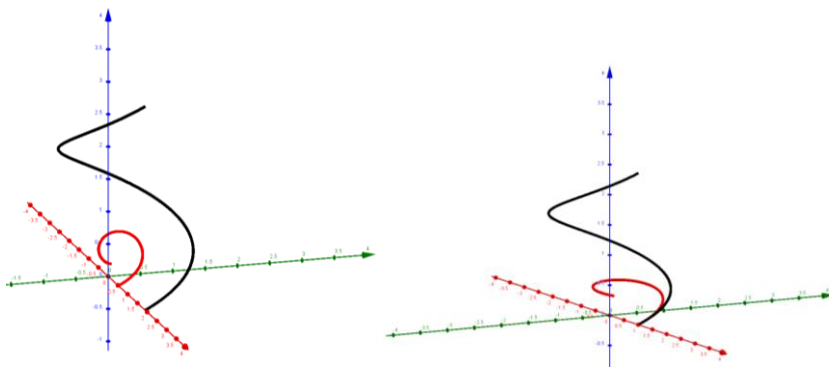


Fig. 7: Inverse curves to spirals

3 Conclusion

Curves inverse to basic planar curves were presented, based on Minkowski point set operations defined according to Minkowski geometric algebra of complex sets. Examples of pairs of inverse curves are presented for line, circle, all conic sections - ellipse, parabola, hyperbola, and spiral, while always the parametric representations of these inverses are derived, and a short analyses of the inverse curves forms with respect to the special positions of the basic curves is provided.

Well-known curves like hippopede of Eudoxus, cissoid of Diocles, limaçon of Pascal or cardioid (epitrochoids), or even lemniscate of Bernoulli and Cartesian ovals might appear as inverse curves to conic sections, if these are in special position to the coordinate system. These interesting findings should be studied further on carefully, in order to find relations between various forms of inverses with respect to transformations of scaling, shifting and probably also rotation of the basic curve in the complex plane. There might be found some connections to Möbius transformations of the complex plane.

An example of possible definition of an inverse to a space curve, namely helix, is presented, too, while more investigations should be paid to this problem.

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Proměny výuky deskriptivní geometrie na VŠB-TUO

Changes in the teaching of descriptive geometry at VSB – Technical University of Ostrava

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Abstract. The paper deals with the changes in the teaching of descriptive geometry at the VSB – Technical University of Ostrava over the last 68 years in the context of technological development and changing requirements of technical disciplines. The main topics are the transition from traditional manual drawing to digital methods of visualization and modeling, as well as changes in the perception of the role of descriptive geometry in engineering education. The paper also analyses which of the topics discussed remain timeless and which have been progressively reduced or replaced, based on their relevance to contemporary engineering disciplines. The insights gained can serve as a starting point for further modernization of the curriculum in order to maintain geometric thinking as one of the cornerstones of engineering training.

Keywords: descriptive geometry, teaching

Klíčová slova: deskriptivní geometrie, výuka

1 Úvod

Deskriptivní geometrie se na VŠB-TUO vyučuje už od jejích počátků, které sahají do poloviny 19. století, kdy se ještě jednalo o báňské učiliště v Příbrami.[2] Katedra matematiky a deskriptivní geometrie vznikla jako samostatný ústav v roce 1899 a dosud zajišťuje výuku matematických předmětů a deskriptivní geometrie.¹

Proměny výuky deskriptivní geometrie budeme sledovat zejména s pomocí učebnic a skript vydaných na VŠB-TUO. Nejstarším skriptem je *Deskriptivní geometrie* autora Václava Štěpánského z roku 1953. Výchozí pro nás je rok 1957, ve kterém vyšlo skriptum *Deskriptivní geometrie II*, jehož autory jsou Václav Štěpánský, Oldřich Hajkr, Jan Hebelka a Josef Láníček. Toto skriptum je nejstarší zdroj, o který se článek opírá.

¹V současnosti spolu s dalšími katedrami matematiky působícími na různých fakultách univerzity.

2 Výuka

Jedním z ukazatelů toho, jak se měnila výuka deskriptivní geometrie, je určitě i časová dotace přidělená předmětu a způsob jeho zakončení. Přehled níže prezentuje situaci na Hornicko-geologické fakultě, kde se deskriptivní geometrie vyučovala po celou dobu sledovaného období. Roky, které nejsou uvedeny se nepodařilo dohledat.

Téměř od počátku sledovaného období až do roku 1976/77 byl počet hodin konstantní, měnil se jen způsob zakončení předmětu.² Od roku 1977/78 ale začal počet hodin postupně klesat, kromě třech okamžiků, kdy došlo k nárůstu. Prvním byl rok 1987/88, kdy se z jednosemestrálního předmětu stal opět dvousemestrální. V roce 1990/91 přibyla navíc ve druhém semestru jedna hodina přednášky, díky čemuž měla výuka stejný rozsah jako v roce 1977/78. Posledním navýšením bylo v roce 1996/97 přidání jedné hodiny přednášky.

Akademický rok	I. semestr	II. semestr
1958/59	3+2 zk	2+2 kl.zap+zk
1959/60 – 1963/34	3+3 zk	2+2 kl.zap+zk
1965/66 – 1966/67	3+3 zk	2+2 zk+pr
1967/68 – 1976/77	3+3 zk	2+2 zk
1977/78	2+2 zk	2+2 zk
1980/81	4+2 zk	
1984/85 – 1985/86	3+3 zk	
1987/88 – 1989/90	2+2 zk	1+2 zap
1990/91 - 1991/92	2+2 zk	2+2 zk
1993/94 – 1995/96	2+3 zk	
1996/97 – 2003/04	3+3 zk	
2004/05 – 2007/08	2+2 zk	
2019/20 – dnes	2+2 zap+zk	

Obr. 1: Časová dotace v jednotlivých akademických rocích spolu se způsobem zakončení.[5]

3 Skripta

Během sledovaného období vyšlo velké množství skript, podklad tohoto článku tvoří 24 z nich.[4]³ Porovnávat mezi sebou jednotlivá skripta by

²zk – zkouška, zap – zápočet, kl.zap – klasifikovaný zápočet, pr – projekt

³Jejich seznam lze na vyžádání zaslat.

bylo nepřehledné a příliš náročné, proto je rozdělíme do skupin, z nichž každá tvoří zdánlivý celek. Pojícím prvkem skupiny je například totožný autorský kolektiv nebo návaznost jednotlivých skript na sebe atp. U každé skupiny uvádíme přibližnou dobu používání skript, resp. rok prvního a posledního vydání některého skripta ze skupiny.

1. 1957 – 1989 V. Štěpánský, O. Hajkr, E. Plocková
2. 1990 – 1998 J. Láníček, E. Plocková
3. 1994 – 2008 Základy DG a KG + Sbírkka řešených příkladů z DG a KG (5 dílů)
4. 2006 – 2013 mdg.vsb.cz
5. 2012 – dnes LMS

První skupina se vyznačuje velkým množstvím vydaných knih, ve kterých se ale témata téměř neměnila. Pro skripta druhé skupiny je charakteristické převzetí témat ze skupiny první a jejich zestručnění. Třetí skupina je ucelenou sadou pěti skript doplněná pěti sbírkami úloh. Čtvrtou skupinu definují materiály šířené především elektronicky, v současnosti shromážděné na uvedené webové adrese.[3] Poslední a současná skupina neobsahuje žádná skripta vydaná klasickým způsobem, ale zahrnuje materiály, které si vyučující tvoří sami a zveřejňují je ve svých elektronických kurzech v LMS.⁴

Časová osa vyznačuje období působnosti jednotlivých skupin skript. Tato doba se postupně zkracuje, což znamená, že rychlost proměňování výuky se zvyšuje.



Obr. 2: Orientační časová osa.

4 Témata

Změny se nejvíce projevují v probíraných tématech, podíváme se tedy na to, která témata autoři do svých skript zařazovali a v jaké míře. Sledujeme vždy jedno téma a porovnáváme to, jak se mu věnovaly jednotlivé skupiny. Zároveň musíme zohlednit, pro které studijní obory byla skripta určena. Ve skupinách 1, 2 a 3 jsou skripta určená pro všechny obory VŠB-TUO, které měly ve studijním plánu deskriptivní geometrii. Výběr konkrétních témat byl pak upřesněn na přednáškách. Ve skupinách 4 a 5 už jsou jednotlivá skripta určena pro různé obory a nedají se tak snadno porovnat, např. proto, že pro některé obory skripta nevznikla.

⁴Learning Management System.

Nyní popíšeme, v jaké míře se v jednotlivých skupinách vyskytovala nejvýznamnější témata a na co kladli autoři důraz. Vše ilustruje přehled na konci odstavce. Kapitulu o vývoji deskriptivní geometrie, stejně jako projektivní geometrii, obsahovaly pouze skripta z první skupiny. Geometrické základy jsou zmiňovány jen v první a druhé skupině a pak ve čtvrté. V prvních dvou skupinách se kladl důraz na axiomatiku, ve čtvrté skupině autoři připomínali také množiny bodů dané vlastnosti a úlohy týkající se trojúhelníků. Zobrazovacím metodám se věnovala skripta v každém období, přičemž je zde třeba upozornit, že pokud v některé skupině nebylo zpracováno určité téma, neznamená to, že by se neprobíralo např. s využitím předchozích materiálů. Naopak lineární perspektiva a kosoúhlé promítání se vyskytují pouze v poslední skupině, což souvisí s výukou deskriptivní geometrie na Stavební fakultě od roku 1997. U kuželoseček se ve všech obdobích probírala elipsa, hyperbola a parabola a v prvních třech obdobích také konstrukce řezů válcových a kuželových ploch. Stálým tématem jsou i křivky, včetně cyklických a plochy, jejichž výběr byl vždy uzpůsoben konkrétnímu typu studia.⁵ Zpočátku se řešilo i osvětlení a sítě ploch, v posledních třech skupinách zůstaly jen konstrukce řezů a průniků.

	1. skupina	2. skupina	3. skupina	4. skupina	5. skupina
Historie, projektivní geometrie	✓	✗	✗	✗	✗
Základy geometrie	axiomy, definice, ∞	axiomy, definice, ∞	✗	axiomy, mbdv, trojúhelníky	✗
Zobrazovací metody	KP, MP, KSP, PAX	KP, MP, KSP, PAX, teorie	KP, MP, PAX	MP, PAX	KP, MP, KSP, PAX, LP
Kuželosečky	E, P, H, řezy, průmět kružnice	E, P, H, řezy, kolineace	E, H, P, řezy	E, H, P	E, H, P
Křivky a plochy	stíny, řezy, průniky	sítě, řezy, průniky	řezy, průniky	řezy, průniky	řezy, průniky

Obr. 3: Přehled témat v jednotlivých skupinách.

5 Závěr

Role deskriptivní geometrie v inženýrské přípravě se proměňuje. Dříve to byl nutný nástroj pro správné zpracování výkresů, dnes se významně podílí na rozvoji prostorové představivosti a umožňuje správné pochopení moderních modelovacích a vizualizačních nástrojů. Studium vlastností geometrických objektů zůstává jejím předmětem stále.[1]

Původní záměr využít poznatky o proměnách výuky k další modernizaci deskriptivní geometrie se ukázal jako obtížně dosažitelný. Sledované

⁵Hornictví, strojnictví, stavebnictví, architektura.

změny nebyly natolik zásadní, aby bylo možné postihnout tendenci vývoje předmětu. Nelze tedy stanovit trend jeho vývoje ani v následujících letech.

Shrňme aspoň poznatky, které se vyskytly při studiu zmiňované problematiky. Časová dotace se průběžně snižuje, na některých školách se od výuky deskriptivní geometrie upouští úplně. Témata se nemění kontinuálně, spíše nárazově mezi jednotlivými skupinami. Intenzita změn narůstá, ale nejsou to změny zásadní, mění se většinou jen dílčí témata, případně se snižuje náročnost. Významná témata jako jsou zobrazovací metody, křivky a plochy a kuželosečky zůstávají beze změn. Do výuky se postupně zapojují nové technologie, např. POV-Ray, GeoGebra a 3D tisk.

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